

RESEARCH ARTICLE

ACTUARIAL ANALYSIS OF CLAIM PER PAYMENT EVENT: THEORETICALLY DEVELOPING ESTIMATION LINK BETWEEN QUANTUM MECHANICS AND INSURANCE MODELING

Ogungbenle, M. G.

Department of Actuarial Science, Faculty of Management Sciences, University of Jos, Nigeria

ABSTRACT

In general, business, underwriting firms usually write policy terms by imposing additional deductible conditions in order to mitigate the risk of loss and discourage frivolous claims by modifying the indemnity payable by the underwriter where loss occurs. Under deductible conditions, underwriting firms will only be liable in a loss event where it becomes apparent that the loss has exceeded the deductible defined in the policy terms. The maximum accumulated number of losses retained by the insured applying deductible policy modifications is usually set as part of the terms and conditions of the policy documents. This paper develops an analytical framework for evaluating the effect of structural properties of dirac-delta on insurance risk variables with deductible clauses. The objective is to obtain models for the excess of loss random variable in a payment event. In order to achieve this and create analytically sound theoretical platform of investigating payment distribution functions, the quantum structure of dirac-delta is first examined in respect of probability density function. The import of adopting the dirac-delta function in this paper lies in its elegance to permit alternative technique to obtain analytically useful models for insurance severity beneficial to both the insured and insurer with particular reference to rate relativity deductible clause. We then obtained insurance excess of loss severity and variance for an arbitrary policy under deductible coverage conditions. As part of our contributions, theorems in respect of loss were stated and proved for underwriters to see reasons for their applications and use in policy underwriting decisions.

Keywords: *Cost per loss, deductible, loss elimination ratio, risk, severity*

DOI. *http://doi.org/10.4038/jsc.v12i2.32*

1. INTRODUCTION TO SINGULARITY FUNCTIONS

In practice, underwriting firms are bound by set targets for claim out-go by using the claims values already recorded and then use it in projecting the frequency of claims in

*Corresponding author: *moyosiolorun@gmail.com* *https://orcid.org/0000-0001-5700-7738*

respect of the uncertain periods. The potential insurance risk associated with allocation of insurance funds necessitates the need for deep actuarial estimation techniques. Insurance funds are usually invested in many debt and market instruments with the goal of generating real returns on investment so as to meet claim obligations criteria and solvency requirements. In [1] we see that the insurance expected loss associated with both claim per payment and claim per loss which form the basis of this paper is responsible to a large extent for a meaningful fraction of the aggregate liability of the underwriting firms and consequently, the loss is associated with claims demand and uncertainties linked to it through which the insurance schemes provoke the underwriter to forfeiture. Usually, underwriters are frequently engaged in the administration of issues relating to the expected claim liability estimation and in order to deal with these issues, underwriting firms will have to apply actuarial techniques of estimation to obtain critical information over the uncertainties on the liabilities in order to ensure decisive actions relating to the expected claim, payout targets and future insurance pricing. Insurance firms are now groping to cope with the current problems of underwriting risk phenomena, regulatory risk trajectories, solvency requirements risk and demand for insurance policies, market share syndrome, risk management, investment risk and claim payment strategies.

It is imperative to note from the foregoing that the issue of satisfying claim payment terms and conditions has become a hydra-headed problem to the underwriters even though the scheme holders want a financial succor in the event of contingencies as defined in the policy terms happen. We infer from [1-2] that the estimation of claim per payment of the insurance policies allow the risk manager to take drastic actions on claim payment devoid of significant error that could evolve out of the problems already enumerated hitherto and consequently it is objective of this paper to apply expectation and singularity potential theory as working tools to model insurance claims in order to estimate the expected claim per payment liabilities in non-life insurance contracts. This is done by developing a link between singularity and actuarial modeling and fix an actuarial model which provides financial arrangements to cover the expenses resulting from the actuarial treatments of insurance losses. The occurrence of a loss event is a necessary condition for advising a claim.

In other to model an insurance loss, the event surrounding the loss must be well defined such that it does not have too much exposure to risk. Correspondence in the analysis of loss coverage usually assumes a rigorous dimension which could conceal the ease of the underlying idea but the dirac-delta function serves to exemplify much of the difficult expression which is the key tool used to deal with actuarial principles involved and to represent magnitude of insurance loss. In this paper, we will apply the dirac-delta function to obtain the expected cost per payment claim severity under deductible conditions and the variance of the cost per payment loss event under the deductible coverage modifications. Note that the effect of the insured’s characteristics on payment amounts functionally depends on the cost sharing arrangements implied by the deductible clauses such that preference for a small deductible could reflect an anti-selective tendency on the part of the scheme-holders.

In [1-2], it is asserted that dirac-delta function exists naturally in the solution of financial problems but it is often applicable in actuarial risk theory and statistical physics to obtain the distribution of an ideal point mass as a function equal to zero everywhere except for zero and whose integral in the entire real line becomes unity. The behavior of the dirac-delta function is such that it is not a real valued function but only a notation $\delta(z)$ which for apparently defined reasons is considered as if it were a function. The dirac-delta function becomes useful in dealing with a defined notation when addressing quantities associated with some kind of infinity and precisely it is deeply related to an eigen function corresponding to an eigenvalue in the continuum that is non-normalizable. The dirac-delta function is treated as the extension of the Kronecker delta function in the event of the continuous variables. Geometrically, the dirac-delta describes the trajectory of a curve whose width approaches nullity and narrow trough roughly approaching infinity while maintaining the area under the curve finite. From its behaviour, it is a real function of z which becomes zero everywhere except on the inside of a small interval of length $\in$ about the centre s_0 but it is extremely bigger in this interval such that its integral over the same interval approaches unity. Following inferences by authors in [1-2], the dirac-delta function plays fundamental roles in actuarial risk theory under the appropriate limit especially in the evaluation of improper integrals involved in probability. In view of observations by authors in [3-4], important applications of dirac-delta were demonstrated in mathematical statistics and probability theory under univariate and multivariate framework. The unit impulse function otherwise called dirac-delta $\delta(z)$ as observed by authors in [1-2;4-5] and which finds applications recently in actuarial risk is a distribution function rather than a true function and it is only defined within an integral on the extended real line. The function is designated generalized real function but does not actually qualify for the characteristics of a real function.

However, in the Schwartz theory of distributions, the function is applied in the evaluation of an integral kernel of some distribution. Following observations by authors in [1;6-8], it is observed that $\delta(z)$ has singularity at a point on the real line where the integral over the extended real line of the product of a function and dirac-delta produces the functional value of the function at that point. Following the definitions by authors in [1;3;9], dirac-delta permits wider spectrum of applications to describe the singularity characteristics of probability distributions used in statistical mechanics especially in quantum theory.

The dirac-delta function is defined in the work of the authors in [1;3;8-11] as follows

$$\delta_{\eta}(z-\omega)=\delta(z-\omega)=\begin{cases}\infty,\text{if}, z=\omega\\ 0,\text{if}, z\neq\omega\end{cases} \text{ and } \int_{-\infty}^{\infty}\delta(z-\omega)dz=\int_{0}^{\infty}\delta(z-\omega)dz=1, \quad (1)$$

the integral is defined over the extended real line. If $\omega = 0$, we have

$$\delta(z-0) = \begin{cases} \infty, \text{if}, z = 0 \\ 0, \text{if}, z \neq 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(z-0)\,dx = 1, \qquad (2)$$
$$\delta(z-0)\, f(z) = \delta(z-0)\, f(0).$$

Furthermore, a particular situation where the product is often defined is that of an integrable real valued function with a dirac-delta structure as long as the real valued function has been well defined at the points of singularities of the dirac-delta. Considering the behavior of the delta function, this is equal to multiplying it by a real number, the value of the integrable real function at the singular point of the delta function.

$$\delta(z-\omega)\, f(z) = \delta(z-\omega)\, f(\omega) \qquad (3)$$

Borrowing from continuous time finance, we can apply complex representation to define dirac-delta function

$$\lim_{A\to\infty}\left[\int_{-A}^{A} e^{ihz}\,dh\right] = \lim_{A\to\infty}\left[\frac{e^{ihz} - e^{-ihz}}{iz}\right] = 2\pi \lim_{A\to\infty}\left[\frac{\sin Az}{\pi z}\right] \qquad (4)$$

As A becomes large,

$$\frac{\sin Az}{\pi z} \to \delta(z) \qquad (5)$$

$$\lim_{A\to\infty}\left[\int_{-A}^{A} e^{ihz}\,dh\right] = 2\pi\delta(z) \Rightarrow \delta(z) = \frac{1}{2\pi}\lim_{A\to\infty}\left[\int_{-A}^{A} e^{ihz}\,dh\right] \qquad (6)$$

As z becomes very large without bounds, we have

$$\int_{-\infty}^{\infty} \frac{\sin Az}{\pi z}\,dz \to 1 \text{ and } \int_{-\infty}^{\infty}\left(\frac{\sin Az}{\pi z}\right) f(z)\,dz \to f(0) \qquad (7)$$

But when as z approaches zero becoming smaller we have

$$\frac{\sin Az}{\pi z} = \frac{Az}{\pi z} = \frac{A}{\pi} \qquad (8)$$

In this section, we use second order differential equation to explain how direct-delta function evolves, as most problems in actuarial risk literature encounter derivation of models for general insurance and casualty. It is on this basis that we use singularity functions to investigate the behavior of actuarial density functions to enable us obtain models applicable in general insurance business.

Recall in the work of the author in [1] that, the linear equation, $b_1 z'' + a_2 z' + a_3 z = f(s)$, is deeply rooted in many areas of actuarial discipline, especially in financial engineering where it has been used to analyze the term structure and varying time parameters of interest rates by setting the forcing function $f(s)=0$ and further assuming that the homogeneous differential equation $b_1 z'' + b_2 z' + b_3 z = 0$ has equal real roots with constant co-efficient $b_i, i = 1,2,3$. Following definitions by authors in [1;5;8;9;11], one of the most simplest but striking application of integral transform occurs in the treatment of linear differential equations with jump discontinuities or discontinuous forcing functions especially in the analysis of circuit problems and mechanical vibrations.

Recall the definitions by the author in [1] that in the second order differential equation above, $f(s)$ is a measure of forcing term and the total area under the curve

$$\int_{-\infty}^{\infty} f(s)ds = \lim_{a\to\infty}\int_{-a}^{a} f(s)ds \tag{9}$$

is the impulsive force.
We define the function

$$\delta_\eta(s-s_0) = \begin{cases} \dfrac{1}{2\eta}, \text{if}, s_0-\eta < s < s_0+\eta \\ 0, \text{if, otherwise} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta_\eta(s-s_0)ds = 1 \tag{10}$$

where $\eta>0$

$$\int_{s_0-\eta}^{s_0+\eta} \delta_\eta(s-s_0)f(s)ds = \int_{s_0-\eta}^{s_0+\eta} \frac{1}{2\eta}f(s)ds = (s_0+\eta-s_0+\eta)\frac{1}{2\eta}f(\bar{s}) \tag{11}$$

$$\int_{s_0-\eta}^{s_0+\eta} \frac{1}{2\eta}f(s)ds = \frac{2\eta}{2\eta}f(\bar{s}) = f(\bar{s}), \text{ using the mean value theorem} \tag{12}$$

$$\eta\to 0, \delta_\eta(s-s_0)\to\delta(s-s_0), \bar{s}\to s_0 \tag{12a}$$

$$\lim_{\eta\to 0}\left[\int_{-\infty}^{\infty}\delta_\eta(s-s_0)f(s)ds\right] = \lim_{\eta\to 0} f(\bar{s}) = \int_{-\infty}^{\infty}\lim_{x\to\infty}\left[\delta_\eta(s-s_0)f(s)\right]ds \tag{13}$$

Since $f(s)$ is well behaved having a unique value at every point in its domain

$$\int_{-\infty}^{\infty}\delta(s-s_0)f(s)ds = f(s_0) \tag{14}$$

and setting $s_0 = 0$, then from equation (15) the dirac-delta function becomes valid in an interval when a rule that integrates its product with another continuous function is assigned hence

$$\int_{-\infty}^{\infty}\delta(s-0)f(s)ds = f(0) \tag{15}$$

Furthermore, following the definition of the author in [12],

$$\delta(\beta-\alpha) = \int \delta(s-\alpha)\delta(\beta-s)ds \tag{16}$$

The integral value 1 and the limiting value 0 both define the value of dirac-delta function δ which has a value 1 when $s=0$ and 0 if otherwise. $if, \int_{-\infty}^{\infty}\delta(s-s_0)f(s)ds \approx f(\bar{s})$, $\delta(s-s_0)$ is the kernel of the integral transform describing the dimensions of a rectangular paralleliped of length 2η and height $\frac{1}{2\eta}$ and centered at s_0 so that the area of the paralleliped will be 1. $\delta(s-s_0)$ isolates the real value of $f(s)$ at some prescribed point s_0 by the normalizing property of dirac-delta function,

$$\delta(s)=\delta(-s)\text{ and }\delta(s-\bar{s})=\delta(\bar{s}-s) \tag{17}$$

$$\int_{-\infty}^{\infty}\delta(s-\bar{s})ds=\int_{-\infty}^{\infty}\delta(s-\bar{s})ds=1\Rightarrow\int_{-\infty}^{\infty}\delta(t)dt=1,\text{ when } t=(s-\bar{s}) \tag{18}$$

Following the work of the author in [11], we let $f(z)$ be a function on which shift operator is defined as

$$E_{\Delta}f(z)=f(z+\Delta)\Rightarrow E_{-\Delta}f(z)=f(z-\Delta) \tag{19}$$

Define the function $G(\xi)=\int_{-\infty}^{\infty}f(\xi)\xi(z)dz$ where $\xi(z)$ is taken from the space of test function. If we assume that the functional $G[\xi]$ corresponds to $f(z)$, then $E_{\Delta}G[\xi]$ corresponds to $E_{\Delta}f(z)$, therefore

$$E_{\Delta}G[\xi]=\int_{-\infty}^{\infty}f(z+\Delta)\xi(z)dz=\int_{-\infty}^{\infty}f(z)\xi(z-\Delta)dz \tag{20}$$

where $E_{\Delta}G[\xi]=G[E_{-\Delta}\xi]$, if we invoke this definition on the dirac-delta function, then we have,

$$E_{\Delta}\delta(\xi)=\int_{-\infty}^{\infty}\delta(z+\Delta)\xi(z)dz=\delta[\xi(z-\Delta)]=\xi(-\Delta) \tag{21}$$

2.0 APPLICATION OF DIRAC-DELTA TO PROBABILITY DENSITY FUNCTIONS

In this section the goal is to test dirac-delta function on arbitrary random risk, to enable us apply it on excess of loss random variable. In view of the work of the authors in [13-14], the function $f(z)$ defines the final pay-off to a unit linked insurance which is maturing at time z. Consider a case where the origin has been shifted from zero to another point s_0, we have

$$z_1 < s_0 < z_2, \int_{z_1}^{z_2} \delta(z - s_0) f(z) dz = \int_{z_1}^{z_2} \delta(z - s_0) f(s_0) dz = f(s_0) \int_{z_1}^{z_2} \delta(z - s_0) dz \qquad (22)$$

$$\int_{z_1}^{z_2} \delta(z - s_0) f(z) dz = f(s_0) \times 1 = \partial(\delta(z)) = f(s_0), \qquad (23)$$

$\partial(.)$ is the Laplace transform of $\delta(z)$.

The point $z = s_0$ is a contribution of the integral in equation (22), that is the first term of the Taylor's series expansion of $f(z)$ at the point $z = s_0$ and which vanishes at all other functional values. Again, substituting $s_0 = 0$, we have

$$\int_{z_1}^{z_2} \delta(z - 0) f(z) dz = f(0) \times 1 = f(0) \times 1 = f(0) \qquad (24)$$

In view of the work of the author in [2], if $H(z)$ is the unit step function,
then

$$\delta(z - k) = \frac{dH(z - k)}{dz} \qquad (25)$$

Now $F_Z(z)$ is the distribution function of a random risk Z with the property
that

$$\frac{dF_Z(z)}{dz} = f_Z(z) \qquad (26)$$

Define $F_Z(z) = \sum_{z_i \in \Omega_Z} P(z_i) H(z - z_i)$, where Ω_Z is the support of Z. By the above property,

$$\frac{dF_Z(z)}{dz} = \frac{d}{dz}\left[\sum_{z_i \in \Omega_Z} P(z_i) H(z - z_i)\right] = \left[\sum_{z_i \in \Omega_Z} P(z_i) \frac{d}{dz}\left[H(z - z_i)\right]\right] \qquad (27)$$

so that the probability density function is obtained as

$$\frac{dF_Z(z)}{dz} = f_Z(z) = \sum_{z_i \in \Omega_Z} P(z_i) \delta(z - z_i) \qquad (28)$$

where $\Omega_Z = \{z_i\}\ i = 1, 2, 3, \ldots$ and $P(z_i)$ are the mass points. Since the finite moments of the density function exists, then the average value of the random risk Z can be computed using the moments.

$$\langle Z \rangle = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_{-\infty}^{\infty} \left[\sum_{z_i \in \Omega_Z} P(z_i) \delta(z - z_i)\right] z dz \qquad (29)$$

$$\langle Z\rangle=\int_{-\infty}^{\infty}\left[\sum_{z_i\in\Omega_Z}P(z_i)\delta(z-z_i)\right]zdz=\left[\sum_{z_i\in\Omega_Z}P(z_i)\int_{-\infty}^{\infty}z\delta(z-z_i)dz\right], \quad (30)$$

since $f_Z(z)=z$.

$$\langle Z\rangle=\int_{-\infty}^{\infty}\left[\sum_{z\in\Omega_Z}P(z_i)\delta(z-z_i)\right]zdz=\left[\sum_{z_i\in\Omega_Z}z_iP(z_i)\right]. \quad (31)$$

$$\langle Z^2\rangle=\int_{-\infty}^{\infty}z^2f_Z(z)dz=\int_{-\infty}^{\infty}\left[\sum_{z_i\in\Omega_Z}P(z_i)\delta(z-z_i)\right]z^2dz. \quad (32)$$

$$\langle Z^2\rangle=\left[\sum_{z_i\in\Omega_Z}P(z_i)\int_{-\infty}^{\infty}z^2\delta(z-z_i)dz\right]=\sum_{z_i\in\Omega_Z}z_i^{\,2}P(z_i). \quad (33)$$

$$Var(Z)=\langle Z^2\rangle-\langle Z\rangle\langle Z\rangle=\left[\sum_{z_i\in\Omega_Z}z_i^{\,2}P(z_i)\right]-\left[\sum_{z_i\in\Omega_Z}z_iP(z_i)\right]^2. \quad (34)$$

Hence the first two moments and variance are well defined. Furthermore, we can use the following approximation.

$$\int_{-\infty}^{\infty}\delta(z-z_1)f_Z(z)dz=\lim_{z_2\to z_1}\left[\frac{\int_{z_1}^{z_2}\delta(z)f_Z(z)dz}{z_2-z_1}\right]=\lim_{z_2\to z_1}\left[\frac{f_Z\left(\frac{z_2+z_1}{2}\right)(z_2-z_1)}{z_2-z_1}\right] \quad (35)$$

$$\int_{-\infty}^{\infty}\delta(z-z_1)f_Z(z)dz=\lim_{z_2\to z_1}\left[f_Z\left(\frac{z_2+z_1}{2}\right)\right]=f_Z(z_1) \quad (36)$$

and dirac-delta function is a useful technique in this kind of estimation.

2.1 Underwriting control measures

In view of the work of the authors in [13-20], we observe that as a result of the consequences of the implicit cost such as moral hazard, underwriters have come up with loss control techniques to off-set these hidden costs from insurance contracts and discourage frivolous claims.

Let $a(s)$ define the value of a non-life insurance portfolio at a time s. The amount of loss in the interval

$(s,s+\delta s)$ is given by $a(s)-a(s+\delta s)=l(s,s+\delta s)$; (37)

$$\frac{a(s)-a(s+\delta s)}{\delta s}=\frac{l(s,s+\delta s)}{\delta s}\Rightarrow a'(s)=\frac{-l(s,s+\delta s)}{\delta s}. \quad (38)$$

Thus, if at times $\tau<s<t$, then

$$a(s)=a\left(\frac{t-s}{t-\tau}(\tau)+\frac{s-\tau}{t-\tau}(t)\right)\geq\frac{t-s}{t-\tau}a(\tau)+\frac{s-\tau}{t-\tau}a(t) \qquad (39)$$

$$(t-s)a(\tau)+(s-\tau)a(t)\leq a(s)(t-\tau) \qquad (40)$$

$$\frac{a(s)-a(\tau)}{s-\tau}\geq\frac{a(t)-a(\tau)}{t-\tau}\geq\frac{a(t)-a(s)}{t-s}\Rightarrow a'(\tau)>a'(s) \qquad (41)$$

implying the differentiability of $a(s)$. If the unit time is being discretized as

$$s_k=k\delta s\,,\ k\in\{0\}\cup Z^+ \qquad (42)$$

then

$$l(s_k,s_{k+1})=a(s_k)-a(s_{k+1})=a(k\delta s)-a((k+1)\delta s) \qquad (43)$$

2.2 Theorem 1: Suppose $a(s)$ is a portfolio of non-life insurance policies and assume the scheme-holder incurs losses at a rate $a'(s)$ such that $a'(s)=-\alpha a(s)$ where α is constant. Let $\alpha\delta s$ define the probability that a policy holder incurs losses in an infinitesimal time of δs and define $\theta(s)$ to be the probability that the scheme holder does not incur losses at time s, then for any $\xi\in(s,s+\delta s)$, $h'(s)=-\alpha h(\xi)$ where h is a real probability density function

Proof: By definition $\theta(s+\delta s)$ is the probability that the scheme holder does not incur losses at time $s+\delta s$ given that he has not incurred losses at time s and consequently

$$\theta(s+\delta s)=(1-\alpha\delta s)\theta(s) \qquad (44)$$

which implies

$$\theta(s+\delta s)=\theta(s)-\alpha\delta s\theta(s) \qquad (45)$$

$$\frac{\theta(s+\delta s)-\theta(s)}{\delta s}=-\alpha\theta(s) \qquad (46)$$

$$\theta'(s)=-\alpha\theta(s) \qquad (47)$$

this defines an exponential solution pattern as observed in the conclusion by the author in [21] and this informs why we apply exponential distribution in our analysis. Then by definition, it is clear that

$$\theta'(s)=-h(s),\ \theta(s)=\frac{h(s)}{\alpha} \qquad (47a)$$

If $h(s)$ is the probability density function of incurring losses at time s, then

$$\frac{d\theta(s)}{ds}=-h(s) \qquad (48)$$

$$1-\theta(s+\delta s)-(1-\theta(s))=-\int_{s}^{s+\delta s}\frac{d\theta(\zeta)}{d\zeta}d\zeta=-\int_{s}^{s+\delta s}\theta'(\zeta)d\zeta= \\ \int_{s}^{s+\delta s}h(\zeta)d\zeta=\theta(s)-\theta(s+\delta s) \tag{49}$$

$$\alpha\int_{s}^{s+\delta s}h(\zeta)d\zeta=(h(s)-h(s+\delta s)) \tag{49a}$$

By the mean value theorem,

$$\int_{s}^{s+\delta s}h(\zeta)d\zeta=(\delta s)h(\xi);\ \xi\in(s,s+\delta s) \tag{50}$$

$$\alpha(\delta s)h(\xi)=(h(s)-h(s+\delta s)) \tag{50a}$$

$$h'(s)=-\alpha h(\xi) \tag{50b}$$

The distribution of the random loss in (37) to the underwriting firm is enumerated below Z assumes the loss incurred in a loss event when there is no deductible but does not functionally depend on deductible and Z_L defines the amount incurred in a loss event under deductible while Z_p is the cost per payment in a payment event under deductible modifications. The loss event describes the condition of a loss but payment event is a condition where an underwriter incurs a fraction of a loss or wholly liable to pay everything. In practice, insurance data are only available on incurred payments. Following the work of the authors in [19-20; 22-25] when $Z<c$, the underwriter will repudiate claim advised except on ex-gratia basis to boost its goodwill and information content on losses will not be available thereby creating problems in insurance analysis. However, under deductible terms and conditions, the excess of loss random variable is only captured to the extent that the information content on Z_L is truncated.

As observed in the definitions by authors in [13;24;26-30], deductibles describe intermediate insurance transfer technique between total loss transfer and self-insurance to an underwriter. It has been applied to arouse interest of a few medium-sized employers but the rationale behind the attractiveness does not usually fall in line with the aims of insurance regulators and issues are usually raised by workers' union generally which need to be resolved. An insured with a per loss deductible c will repudiate claims whenever the claim of size Z falls short of or equal to the deductible c. However, when the claim value rises above the value c, the underwriter will pay the excess $(Z-c)$ The amount of loss covered by the underwriter and paid out as claim size in the loss event is defined in [1-2] by

$$Z_{\mathrm{L}}=\begin{cases}0 & \mathrm{Z}\le\mathrm{c}\\ Z-c & \mathrm{Z>c}\end{cases} \tag{51}$$

$$Z_{\mathrm{L}}, 0 \le Z_{\mathrm{L}} \le z \tag{52}$$

$$Z_{\mathrm{L}} = (Z - \mathrm{c})_{+} \text{ where } Z_{+} = \begin{cases} 0 & Z \le 0 \\ Z & Z > 0 \end{cases} \tag{53}$$

$$Z_{+} = \begin{cases} Z & for \quad Z \le c \\ c & for \quad Z > c \end{cases} \tag{54}$$

This is the amount retained by the insured

$$\langle Z_{\mathrm{L}} \rangle = \langle (Z - \mathrm{c})_{+} \rangle = \int_{0}^{\infty} \Pr(Z > \mathrm{c} + z) dz \tag{55}$$

If $\Pr(Z_{\mathrm{L}} = 0) = \mathrm{F}_{Z}(\mathrm{c})$ (56)

Then Z_{L} has a probability mass point at zero of $\mathrm{F}_{Z}(c)$ and hence

$$\mathrm{f}_{Z_{\mathrm{L}}}(Z) = \mathrm{f}_{Z}(z + \mathrm{c}) \text{ for } z > 0 \tag{57}$$

The expected value function allows us to assess which losses from the risks, the insurance firm will bear in quantitative terms The random variable $(Z-c)_{+}$ describes the amount by which Z is greater than the threshold ceiling c,

$$(Z - c)_{+} = \int_{c}^{\infty} S_{Z}(\chi) d\chi \tag{58}$$

where $(\chi)_{+} = \max(\chi, 0)$. Furthermore,

$$\lim_{c \to -\infty} \left[(Z - c)_{+} + c \right] = \int_{c}^{\infty} S_{Z}(\chi) d\chi = \lim_{c \to -\infty} E(\max(Z, c)) = \langle Z \rangle \tag{59}$$

$$\langle Z \rangle = \int_{0}^{\infty} z f_{Z}(z) dz, \; f_{Z}(z) = \frac{\Pr(z < Z < z + \delta z)}{\delta z} \tag{60}$$

where $\langle . \rangle$ is the average value function.

We assume the existence of a positive differentiable non decreasing function of finite integral

$$\int_{0}^{\infty} \varphi(z) dF(z) < \infty \tag{61}$$

$$\langle \varphi(z) \rangle = \int_{0}^{\infty} \varphi(z) dF_{Z}(z) = -\int_{0}^{\infty} \varphi(z) d\left(\int_{0}^{\infty} f_{Z}(z) dz - F_{Z}(z) \right) \tag{62}$$

$$\langle \varphi(z) \rangle = -\varphi(\infty)\left[1 - F_{Z}(\infty)\right] + \varphi(0)\left[1 - F_{Z}(0)\right] + \int_{0}^{\infty} (1 - F_{Z}(\infty)) d\varphi(z) \tag{63}$$

$$\varphi(0) = 0, \varphi(0)\left[1 - F_{Z}(0)\right] = 0, and, \varphi(\infty)\left[1 - F_{Z}(\infty)\right] = 0, F_{Z}(\infty) = 1 \tag{64}$$

$$\langle \varphi(z) \rangle = \int_0^\infty \left(1 - F_Z(\infty)\right) d\varphi(z) \quad (65)$$

Suppose $p(z)$ is the payment function, then

$$\langle p(z) \rangle = \int_0^\infty \left(1 - F(z)\right) dp(z),\ p(0) = 0 \quad (66)$$

If $\langle p(z) \rangle = \Omega$, then the mean loss implies that

$$\Omega = \int_0^\infty \left(1 - p(z)\right) dp(z) \quad (67)$$

By definition,

$$\int_c^\infty \left(1 - F_Z(z)\right) dz = \int_c^\infty S_Z(z)\, dz = \Omega \quad (68)$$

$$F_{Z_L}(z) = F_Z(z + \mathrm{c}),\ z > 0 \quad (69)$$

$$S_{Z_L}(z) = S_Z(z + \mathrm{c}) \quad (70)$$

$$\langle Z_L \rangle = \frac{\langle \max(0, Z - c) \rangle}{\langle z \rangle} = \frac{\langle \min(Z, c) \rangle}{\langle z \rangle} = \int_{-\infty}^{\infty} (z)\, \mathrm{f}_{Z_L}(z) \mathrm{dz} =$$

$$\int_0^\infty (z)\, \mathrm{f}_{z_L}(z) \mathrm{dz} = \int_c^\infty (z - c)\, \mathrm{f}_Z(z) \mathrm{dz} \quad (71)$$

$$\langle Z_L \rangle = \int_c^\infty (z - c)\, dF_Z(z) = (z - c) F_Z(z) |_c^\infty - \int_c^\infty F_Z(z)\, dz = 1 - \int_c^\infty F_Z(z)\, dz \quad (72)$$

3.0 MATERIAL AND METHODS

The dirac-delta technique has been pesented as a novel advanced method to model insurance severity in order to usher in fresh light into the puzzling investigations in actuarial literature that parametric models tend to equally solve when deductible clauses are built into insurance contracts. The conditional excess of loss random variable remains valid provided there is a payment.

$$Z_p = (Z - c) \mid Z > c \quad (73)$$

$$Z_p = Z_L \mid Z_L > 0 \quad (74)$$

$$\langle Z_L \rangle = E(Z_L \mid Z_L = 0) P(Z_L = 0) + E(Z_L \mid Z_L > 0) P(Z_L > 0)$$

$$\langle Z_L \rangle = 0 + E(Z_L \mid Z_L > 0) P(Z_L > 0) = \langle Z_P \rangle P(Z_L > 0) \quad (75)$$

$$F_{Z_L}(z)=F_Z(z+c)\Rightarrow F_{Z_L}(0)=F_Z(c) \tag{75}$$

$$S_{Z_L}(z)=S_Z(z+c)\Rightarrow S_{Z_L}(0)=S_Z(c) \tag{76}$$

$$\langle Z_L\rangle=\langle Z_P\rangle S_{Z_L}(0)=\langle Z_P\rangle S_Z(c) \tag{77}$$

$$\langle Z_P\rangle=\frac{\langle Z_L\rangle}{S_Z(c)}=\frac{\langle \max(0,Z-c)\rangle}{\langle z\rangle S_Z(c)}=\frac{\langle \min(Z,c)\rangle}{\langle z\rangle S_Z(c)} \tag{78}$$

$$f_{Z_p}(z)=\frac{F_Z'(z+c)}{(1-F_Z(c))}=\frac{f_z(z+c)}{(1-F_Z(c))},S_{Z_p}(z)=\frac{S_Z(z+c)}{(1-F_Z(c))} \tag{79}$$

$$\langle Z_P\rangle=\int_0^\infty(z)dF_{z_p}(z)=\int_0^\infty(z)\frac{dF_Z(z+c)}{(1-F_Z(c))}=\int_c^\infty(z-c)\frac{dF_Z(z)}{(1-F_Z(c))}=\frac{\langle z_L\rangle}{(1-F_Z(c))} \tag{80}$$

$$\langle Z_P\rangle=\frac{1}{(1-F_Z(c))}\int_c^\infty(Z-c)\sum_{j=1}^m P_j\delta\left(Z-z_j^*\right)dz=\frac{1}{(1-F_Z(c))}\sum_{j=1}^m P_j\int_c^\infty(Z-c)\delta\left(Z-z_j^*\right)dz$$

$$\langle Z_P\rangle=\frac{\sum_{j=1}^m P_j\left(z_j^*-c\right)}{(1-F_Z(c))}=\frac{\sum_{j=1}^m P_j z_j^*}{(1-F_Z(c))}-\frac{c\sum_{j=1}^m P_j}{(1-F_Z(c))} \tag{81}$$

$$\langle Z_P\rangle=\frac{\sum_{j=1}^m P_j\left(z_j^*-c\right)}{(1-F_Z(c))}=\frac{\sum_{j=1}^m P_j z_j^*}{(1-F_Z(c))}-\frac{c}{(1-F_Z(c))},\sum_{j=1}^m P_j=1 \tag{82}$$

$$\langle Z_P\rangle=\frac{\sum_{j=1}^m P_j\left(z_j^*-c\right)}{\left(1-\int_{-\infty}^c f_Z(t)dt\right)}=\frac{\sum_{j=1}^m P_j z_j^*}{\left(1-\int_{-\infty}^c f_Z(t)dt\right)}-\frac{c}{\left(1-\int_{-\infty}^c f_Z(t)dt\right)} \tag{83}$$

$$\langle Z_P\rangle=\frac{\sum_{j=1}^m P_j\left(z_j^*-c\right)}{\left(1-\int_0^c f_Z(t)dt\right)}=\frac{\sum_{j=1}^m P_j z_j^*}{\left(1-\int_0^c f_Z(t)dt\right)}-\frac{c}{\left(1-\int_0^c f_Z(t)dt\right)} \tag{84}$$

Based on the use of the density function in equation (84), we state and prove the following theorem.

3.1 Theorem 2:

$$\text{Let } Z=\begin{cases}0 & with \quad probility \quad p\\ B & with \quad probability \quad (1-p)\end{cases} \tag{85}$$

then p can be estimated as

$$\text{(i) } p = \left(\frac{F_Z(t) - F_B(t)}{S_B(t)} \right); \quad (86)$$

$$\text{(ii) } q = \frac{f_Z(t)}{f_B(t)}. \quad (87)$$

Proof

Let
$$I_B = \begin{cases} 0 & having \quad probability \quad p \\ 1 & having \quad probability \quad (1-p) \end{cases} \quad (88)$$

be the indicator function of the random loss B.

Hence $Z = BI_B$. Suppose $B > 0$ is the loss event, but if $I_B = 0$, then $Z = 0$.
For any real $t > 0$,

$$\int_{-\infty}^{\infty} f_Z(t)dt = \Pr(t \geq 0) = 1 - \Pr(t < 0) = 1 \quad (86)$$

$$\int_{-\infty}^{t} f_Z(s)ds = \Pr(Z \leq s) = \Pr(I_B = 0)\Pr(Z \leq s \mid I_B = 0) + \Pr(I_B = 1)\Pr(Z \leq s \mid I_B = 1) \quad (87)$$

$$\int_{-\infty}^{t} f_Z(s)dt = \Pr(I_B = 0)\Pr(BI_B \leq s \mid I_B = 0) + \Pr(I_B = 1)\Pr(BI_B \leq s \mid I_B = 1) \quad (88)$$

$$\int_{-\infty}^{t} f_Z(s)ds = (p)\Pr(s \geq 0) + (1-p)\Pr(B \leq s) = p + (1-p)F_B(s) \quad (89)$$

$$\int_{0}^{t} f_Z(s)ds = (p)\Pr(s \geq 0) + (1-p)\Pr(B \leq s) \quad (90)$$

$$F_Z(s) = p \times 1 + (1-p)F_B(s) = p \times (1 - F_B(s)) + F_B(s) = p \times (S_B(s)) + F_B(s) \quad (91)$$

$$\left(\frac{F_Z(s) - F_B(s)}{S_B(s)} \right) = p \quad (92)$$

From equation (91),

$$F_Z'(s) = (1-p)F_B'(s) \Rightarrow f_Z(s) = (1-p)f_B(s) = qf_B(s) \quad (93)$$

$$q = \frac{f_Z(s)}{f_B(s)} \quad (94)$$

3.2 Theorem 3: As a consequence of the application of expected value function, we also state and prove the following. If $Z > 0$ is a random risk and let $\eta(Z)$ be a differentiable function of Z such that $\lim_{Z \to 0} \eta(Z) = 0$, then

$$E\left(\eta\left(Z\right)\right)=\int_{-\infty}^{\infty}\eta'\left(Z\right)\Pr\left(Z>s\right)ds=\int_{0}^{\infty}\eta'\left(Z\right)\Pr\left(Z>s\right)ds \tag{95}$$

Proof:

Let

$$I_{Z>s}=\begin{cases}1 & for \quad Z>s\\ 0 & elswhere\end{cases} \tag{96}$$

Then by definition,

$$E\left(I_{Z>s}\right)=\Pr\left(Z>s\right) \tag{97}$$

$$\int_{-\infty}^{\infty}\eta'\left(s\right)I_{Z>s}ds=\int_{0}^{Z}\frac{d\eta\left(s\right)}{ds}ds=\int_{0}^{Z}d\eta\left(s\right)ds= \\ \eta\left(Z\right)-\eta\left(0\right)=\eta\left(Z\right)-0=\eta\left(Z\right) \tag{100}$$

$$E\left(\int_{-\infty}^{\infty}\eta'\left(s\right)I_{Z>s}ds\right)=\int_{0}^{Z}E\left(\eta'\left(s\right)I_{Z>s}\right)ds=\int_{0}^{Z}\left(\eta'\left(s\right)\right)E\left(I_{Z>s}\right)ds=E\left(\eta\left(Z\right)\right) \tag{101}$$

$$E\left(\int_{-\infty}^{\infty}\eta'\left(s\right)I_{Z>s}ds\right)=\int_{0}^{Z}\left(\eta'\left(s\right)\right)\Pr\left(Z>s\right)ds \tag{102}$$

$$E\left(\int_{0}^{\infty}\eta'\left(s\right)I_{Z>s}ds\right)=\int_{0}^{Z}\left(\eta'\left(s\right)\right)\Pr\left(Z>s\right)ds=E\left(\eta\left(Z\right)\right) \tag{103}$$

The distribution of loss above shows the probability of a defined magnitude coupled with the probability of a particular loss exceeding or falling under a certain loss. The distribution of loss can then be employed to calculate both the expected loss excess of the deductible amount and the expected proportion of total losses at a defined ceiling. The severity model represents the actuarial technique of achieving the expected size of claims which an insurance firm may likely experience in a given period and the cost of average claim. As reported in the work of the author in [25], we see that in the severity technique, past data profile is used to model the estimated average size of claims and the average cost per claim. A high frequency of claims may indicate that the underwriting firm expects a large number of claims. Insurance experts apply advanced actuarial models to determine the probability that insurance firm will pay out a claim and summarize insurance data set which will be subsequently needed and properly interpreted for underwriting decision process. Appraising actuarial model to compute rate differentials may not be immediately apparent since policy holder behavior may influence the frequency the number of events and the severity of the events. Thus, the magnitude by which loss is eliminated is the reduction in the loss incurred.

$$LE\left(z\right)=\left\langle z\right\rangle-\left\langle z_L\right\rangle . \tag{104}$$

The loss elimination ratio:

$$LER=\frac{\left\langle z\right\rangle-\left\langle z_L\right\rangle}{\left\langle z\right\rangle}=\frac{\left\langle z\right\rangle-\left\langle \max\left(0,Z-c\right)\right\rangle}{\left\langle z\right\rangle}=\frac{\left\langle \min\left(Z,c\right)\right\rangle}{\left\langle z\right\rangle} \tag{105}$$

computes the ratio of the reduction in the expected mean loss of an underwriter writing a contract due to defined deductible conditions c to the expected loss of same underwriter writing a full coverage

$$\langle \min(\theta, c) \rangle = \int_0^c \left(1 - F_\theta(z)\right) dz \tag{106}$$

3.3 Data analysis

In general insurance practice, *data on deductibles* is usually unavailable because they are claims which are only borne by individual scheme holder and moreover because of the confidentiality of insurance data base. Instead of the raw deductible data, we obtained rate relativity on deductible through a non-life insurance *agent* operating in property insurance market at Lagos. In order to present logical arguments, we solve the following standard empirical problem by considering an insured risk Y with unit sum insured of insurance cover with specified deductibles C under an *assumption* of exponential distributions $0.1 \le C \le 1, Z \sim EXP(\alpha), \alpha = 1$. For ease of computation, we consider the exponential distribution.

3.4 Exponential distribution

$$S_Z(C) = e^{-\alpha C}, g_Z(C) = \alpha e^{-\alpha C}, H_Z(y) = \frac{g_Z(C)}{S_Z(C)} \tag{107}$$

$$\langle Z_L \rangle = \langle (z - 0.15)_+ \rangle = \int_{0.15}^{\infty} e^{-z} dZ = e^{-0.15} = 0.86071, \tag{108}$$

$$S_Z(0.15) = e^{-0.15} = 0.86071 \Rightarrow \frac{\langle Z_L \rangle}{S_Z(C)} = \langle Z_P \rangle_{\exp} = \int_0^{\infty} e^{-z} dz = 1 \tag{109}$$

Now, $\langle Z_p \rangle = \int_0^{\infty} g_Z(z) dz = 1$, hence, we can see that $\langle Z_L \rangle < \langle Z_p \rangle$ that is the cost per loss amount is less than the cost per payment amount. The loss eliminated (*LE*) and loss elimination ratio (*LER*) are as given below

$$LE(z) = \frac{1}{\alpha} - \langle Z_L \rangle = 1 - \langle Z_L \rangle, LER(z) = \frac{\langle Z \rangle - \langle Z_L \rangle}{\langle Z \rangle} \tag{110}$$

$$LR(z) = \frac{1}{\alpha} - E(Z_L) = 1 - 0.86071 = 0.13929$$

TABLE 1: Relativity of Deductible and Loss Elimination Ratio for Exponentially Distributed Claim

Deductible C	**Cost Per Loss** Z_L	**Loss Ratio (LR)**	**Loss Elimination ratio(LER)**	**Change in LER**
0.1	0.904837	0.095163	0.095163	-
0.15	0.860708	0.139292	0.139292	0.0441
0.2	0.818731	0.181269	0.181269	**0.042**
0.25	0.778801	0.221199	0.221199	0.0399
0.3	0.740818	0.259182	0.259182	0.038
0.35	0.704688	0.295312	0.295312	0.0361
0.4	0.67032	0.32968	0.32968	0.0344
0.45	0.637628	0.362372	0.362372	0.0327
0.5	0.606531	0.393469	0.393469	0.0311
0.55	0.57695	0.42305	0.42305	0.0296
0.6	0.548812	0.451188	0.451188	0.0281
0.65	0.522046	0.477954	0.477954	0.0268
0.7	0.496585	0.503415	0.503415	0.0255
0.75	0.472367	0.527633	0.527633	0.0242
0.8	0.449329	0.550671	0.550671	0.023
0.85	0.427415	0.572585	0.572585	0.0219
0.9	0.40657	0.59343	0.59343	0.0208
0.95	0.386741	0.613259	0.613259	0.0198
1	0.367879	0.632121	0.632121	0.0189

From the table 1 above, we observe that as the deductible increases, the loss eliminated also increases and consequently, the ratio of the loss eliminated seems directly proportional to the deductibles but from the last column and below the indicator level of $\mathbf{4.2}\%$, it seems high deductibles would not offer a reasonable fraction of the eliminated loss due to the underwriter. Consequently, column 4 represents the ratio of a reduction in the expected loss for an underwriting firm which writes a scheme with a deductible clause or with a policy limit imposed on the expected loss where the firm provides full insurance cover.

The considerations for deductible are quite different depending on policy terms & conditions and on the risk preferences of the scheme holder. The underwriter may consider to apply the above hypothetical table of deductible relativity and use it as a guide to confirm if the deductible proposed by the scheme holder could offer a reasonable level of losses eliminated to the underwriter. However, the insurance firm should be cautious as high deductible is not financially ethical for the insured because they would bear a higher percentage of the losses arising from the insured peril. However, high deductibles may be imposed because of underwriting cost saving conditions, loss control motivations and the burden of falling residuary insurance market. Irrespective of the operating deductible, the goal is to make the scheme holder risk-conscious since he would pay the proportion of the total loss. However, provided that large number of losses is lower than the deductible, the administrative costs incurred by the underwriter to offset liabilities and maintain solvency will drop hence the premium payable by the policy holder will decline compared to full coverage conditions.

From the computations above, we see that a policy limit at $C = 0.150$ for instance shows a loss elimination ratio of 0.139292 meaning that roughly 13.92% of losses incurred will be eliminated by introducing a modification of 0.15. We observe further that the cost per loss values in column 2 are strictly less than unity which is the value of cost per payment in a payment event, implying that cost per loss will always be strictly less than the cost per payment, hence $\langle Z_L \rangle < \langle Z_p \rangle = 1$. Theoretically, it is expected that the net present value of cash in-flows to the underwriting firm will exceed the underwriting income in the event that the investment income cash flow is assumed. This is because premium charged is paid to the underwriter at the time when scheme is incepted, but claims are assumed to be paid in the long run and it is therefore reasonable to assume that the underwriter would have earned return on invested premium.

4. RESULTS AND DISCUSSION

Using the earlier definition of $\langle Z_p \rangle$, we find that

$$\langle Z^2{}_{\mathrm{p}} \rangle = \frac{1}{\left(1 - F_Z(c)\right)} \int_{-\infty}^{\infty} (z - \mathrm{c})^2 \, dF_{Z_L} \, \mathrm{dz} \tag{111}$$

Because density is only defined on the real line, we integrate from zero to infinity

$$\langle Z^2{}_{\mathrm{p}} \rangle = \int_0^{\infty} (z - \mathrm{c})^2 \, \mathrm{f}_{Z_L}(z) \mathrm{dz} \tag{112}$$

But by the definition of deductible, we integrate from c to infinity

$$\langle Z^2{}_{\mathrm{p}} \rangle = \frac{1}{\left(1 - F_Z(c)\right)} \int_{\mathrm{c}}^{\infty} (z - \mathrm{c})^2 \sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{P_j} \delta\left(z - \mathrm{z_j^*}\right) \mathrm{dz} \tag{113}$$

$$\langle Z^2{}_{\mathrm{p}} \rangle = \frac{1}{\left(1 - F_Z(c)\right)} \sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{P_j} \int_{\mathrm{c}}^{\infty} (z - \mathrm{c})^2 \, \delta\left(z - \mathrm{z_j^*}\right) \mathrm{dz} \tag{114}$$

$$\langle Z^2{}_{\mathrm{p}} \rangle = \frac{1}{\left(1 - F_Z(c)\right)} \sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{P_j} \left(z_{\mathrm{j}}^* - \mathrm{c}\right)^2 = \frac{\sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{P_j} \left(z_{\mathrm{j}}^*\right)^2}{\left(1 - F_Z(c)\right)} - \frac{2\mathrm{c} \sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{P_j z_j^*}}{\left(1 - F_Z(c)\right)} + \frac{\sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{P_j c^2}}{\left(1 - F_Z(c)\right)} \tag{115}$$

$$\left\langle Z^2{}_{\mathrm{p}}\right\rangle=\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)}-\frac{2\mathrm{c}\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}+\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)} \tag{116}$$

$$\operatorname{Var}\left(Z_{\mathrm{p}}\right)=\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)}-\frac{2\mathrm{c}\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}+\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)}-\left(\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}-\frac{\mathrm{c}}{\left(1-F_Z(c)\right)}\right)^2 \tag{117}$$

$$\operatorname{Var}\left(Z_{\mathrm{p}}\right)=\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)}-\frac{2\mathrm{c}\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}+\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)}-$$
$$\left(\left(\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}\right)^2+\left(\frac{\mathrm{c}}{\left(1-F_Z(c)\right)}\right)^2-2\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}\left(\frac{\mathrm{c}}{\left(1-F_Z(c)\right)}\right)\right) \tag{118}$$

$$\operatorname{Var}\left(Z_{\mathrm{p}}\right)=\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)}-\frac{2\mathrm{c}\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}+\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)}-\frac{\left(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)^2}-$$
$$\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)^2}+\frac{2c\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)^2} \tag{119}$$

$$\operatorname{Var}\left(Z_{\mathrm{p}}\right)=\frac{\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)}-\frac{2\mathrm{c}\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)}+\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)}-\frac{\left(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)^2}- \tag{120}$$
$$\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)^2}+\frac{2c\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)^2}$$

$$\operatorname{Var}\left(Z_{\mathrm{p}}\right)=\frac{\left(1-F_Z(c)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)^2}-\frac{2\mathrm{c}\left(1-F_Z(c)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)^2}+\frac{\mathrm{c}^2\left(1-F_Z(c)\right)}{\left(1-F_Z(c)\right)^2}-\frac{\left(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}\right)^2}{\left(1-F_Z(c)\right)^2}$$
$$-\frac{\mathrm{c}^2}{\left(1-F_Z(c)\right)^2}+\frac{2c\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z(c)\right)^2} \tag{120a}$$

$$\mathrm{Var}\left(Z_{\mathrm{p}}\right)=\frac{\left(1-F_Z\left(c\right)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^{2}}{\left(1-F_Z\left(c\right)\right)^{2}}-\frac{\left(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}\right)^{2}}{\left(1-F_Z\left(c\right)\right)^{2}}-\frac{2\mathrm{c}\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z\left(c\right)\right)^{2}}+\frac{2c\left(F_Z\left(c\right)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z\left(c\right)\right)^{2}}$$

$$+\frac{\mathrm{c}^{2}}{\left(1-F_Z\left(c\right)\right)^{2}}-\frac{\mathrm{c}^{2}\left(F_Z\left(c\right)\right)}{\left(1-F_Z\left(c\right)\right)^{2}}-\frac{\mathrm{c}^{2}}{\left(1-F_Z\left(c\right)\right)^{2}}+\frac{2c\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z\left(c\right)\right)^{2}} \quad (121)$$

$$\mathrm{Var}\left(Z_{\mathrm{p}}\right)=\frac{\left(1-F_Z\left(c\right)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^{2}}{\left(1-F_Z\left(c\right)\right)^{2}}-\frac{\left(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}\right)^{2}}{\left(1-F_Z\left(c\right)\right)^{2}}+\frac{2c\left(F_Z\left(c\right)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(1-F_Z\left(c\right)\right)^{2}}-\frac{\mathrm{c}^{2}\left(F_Z\left(c\right)\right)}{\left(1-F_Z\left(c\right)\right)^{2}} \quad (121a)$$

$$\mathrm{Var}\left(Z_{\mathrm{p}}\right)=\frac{\left(1-F_Z\left(c\right)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}\left(z_{\mathrm{j}}^{*}\right)^{2}}{\left(S_Z\left(c\right)\right)^{2}}-\frac{\left(\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}\right)^{2}}{\left(S_Z\left(c\right)\right)^{2}}+\frac{2c\left(F_Z\left(c\right)\right)\sum_{\mathrm{j=1}}^{\mathrm{m}}\mathrm{P_j}z_{\mathrm{j}}^{*}}{\left(S_Z\left(c\right)\right)^{2}}-\frac{\mathrm{c}^{2}\left(F_Z\left(c\right)\right)}{\left(S_Z\left(c\right)\right)^{2}} \quad (122)$$

This variance of the random claim size under the deductible policy contract accounts for the fluctuations of risk indicators and defines the degree of variations of outcome produced from the model. The variance of cost per payment may likely fall within many standard deviations of its severity so that small variance will lead to prectitable probability outcome especially when computing the probability that an insurance firm will make aggregate loss or profit over all its insured schemes. Thus the magnitude by which loss is eliminated (*LE*) is the reduction in the loss incurred

$$LE\left(z\right)=\left\langle z\right\rangle-\left\langle z_L\right\rangle \quad (123)$$

Based on the definitions in equations (58), (65), (66) and (71), if S is the sum insured; $k=\left(1-c\right)$, the coverage level for $0\le k\le 1$; Z the risk insured; then the premium of insurance contract under the defined deductible level chosen by the policy holder could be obtained based on a convex actuarial premium rating function $p\left(k\right)$ as follows.

4.1 Theorem 4: Let h be the premium equation which can be written linearly as a function of k

$$h\left(k\right)-p\left(k_0\right)=p'\left(k_0\right)\left(k-k_0\right)=m\left(k-k_0\right) \quad (124)$$

where

$$m = \left.\frac{dp(k)}{dk}\right|_{k=k_0} \text{ is the gradient of } h \tag{125}$$

then

$$p(k_0) \le p(k_1) + \frac{(k_0 - k_1)}{(k_2 - k_0)}\left(p(k_2) - p(k_0)\right) \tag{126}$$

Proof: From our arguments in equations (37), we can write the premium equation linearly as

$$p(k) - h(k) = p(k) - \left(p(k_0) + m(k - k_0)\right) = p(k) - p(k_0) - m(k - k_0) \tag{127}$$

$$p(k) - h(k) = p(k) - \left(p(k_0) + m(k - k_0)\right) = \left(p'(\theta) - p'(k_0)\right)(k - k_0) \tag{128}$$

for $y_0 < \theta < y$.

If p is convex, then $p' > 0$, therefore it implies that either both $p'(\theta) - p'(k_0) > 0$ and $(k - k_0) > 0$ or both $p'(\theta) - m < 0$ and $(k - k_0) < 0$, hence in either case

$$p(k) \ge h(k) \Rightarrow p(k) - h(k) \ge 0 \tag{129}$$

consequently,

$$p(k) - h(k) = p(k) - \left(p(k_0) + m(k - k_0)\right) \ge 0 \tag{130}$$

again, either

$$\frac{p(k) - p(k_0)}{k - k_0} \le m; k < k_0 \tag{131}$$

or

$$\frac{p(k) - p(k_0)}{k - k_0} \ge m; k > k_0 \tag{132}$$

and consequently, if we have $k_1 < k_0 < k_2$, then

$$\frac{p(k_0) - p(k_1)}{k_0 - k_1} \le \frac{p(k_2) - p(k_0)}{k_2 - k_0} \Rightarrow p(k_0) \le p(k_1) + \frac{(k_0 - k_1)}{(k_2 - k_0)}\left(p(k_2) - p(k_0)\right) \tag{133}$$

Suppose $p(k)$ is an increasing and convex function such that $p'(k) > 0$ and $p''(k) > 0$, then the total premium for the chosen coverage level k can then be $S \times k \times p(k)$. Under the distribution of underlying risk, the premium function above can be defined in the form

$$p(k) = \frac{1}{Sk}\int_0^{S(1-C)} \max(0, Sk - Z)\, dF_Z(z) \tag{134}$$

$$p(k) = \frac{1}{Sk}\int_0^{S(1-C)} \max(0, Sk - Z)\, f_Z(z) dz \tag{135}$$

where $f_Z(z)$ is the probability density function of the insured risk.
We note that

$$\frac{1}{SK}\int_{0}^{S(1-C)} \max(0, Sk - Z) f_Z(z)dz < \int_{-\infty}^{\infty} f_Z(z)dz = 1 \quad (136)$$

$$\int_{0}^{S(1-C)} \max(0, Sk - Z) f_Z(z)dz < Sk \text{, for every } k > 0 \quad (137)$$

and such that the distribution function of the coverage level is

$$F_Z(Sk) = \int_{0}^{S+C-1} f_Z(z)dz \quad (138)$$

CONCLUSION

The technique of computing mean loss subject to contract modifications was investigated where we obtained and compared models for computing amount paid in a loss event and in a payment event. Insurance contracts are modified to achieve typical payment functions such as deductible the effect of which is appraised in this paper. In describing a platform of applying generalized functions to study the behavior of risk functions, the dirac-delta technique has been applied to formulate insurance model regarding claim severities and the variance function. The dirac-delta function has then been successfully applied with the objective of drawing attention to some grey area applications of this function in general business insurance. Part of the motivation for using dirac-delta functions, lies in its elegance to permit alternative technique to obtain analytically useful models for insurance severity. In this paper, the severity of an insurance contract under direct delta function with particular rate relativity deductible clause is obtained but this could lead to a critical issue if the rate relativity deductible is not known or missing. Dirac-delta function approach is technically much more convenient in terms of computational superiority and soundness. After appraising the dynamics governing the severity, we are able to build a loss model by applying the dirac-delta function.

The paper presents a dirac-delta-deductible method in the analysis of severity coverages. Correspondence in the analysis of severity coverage usually assumes a rigorous dimension which could conceal the ease of the underlying idea but the dirac-delta function serves to exemplify much of the difficult expression which is the key tool used to deal with actuarial principles involved and to represent magnitude of insurance loss. In this paper, we have applied the dirac-delta function to obtain: **(i)**The expected cost per payment claim severity under deductible policy of general insurance using first moment as reported in (84) **(ii)**The second moment of cost per payment under deductible policy contract of general insurance as reported in (116) **(iii)**The variance of the cost per payment loss event under the deductible coverage modifications obtained and reported in equation (122).

REFERENCES

[1] Ogungbenle, M.G., Adeyele J. S., and Mesioye A.E. (2020.). Modeling moments of insurance claim size under dirac-delta function. *Daffodil International University Journal of Science and Technology*, ***15***(1), 43-51.

[2] Ogungbenle, M.G. (2021a). A theoretical Cauchy-Bunyakovsky-Schwartz inequality estimation of upper bound for the expected claim per loss payment in non-life insurance business. *Journal of New Frontiers in Mathematics and Statistics*, ***2***(2), 1-19.

[3] Mohammed H.R. (2011). Dirac's function and its application in solving some problems in mathematics. *Journal of Al-Qadisiya for Computer Science and Mathematics* ***3***, 1-15.

[4] DeLyra J. L. (2018). Complex analysis of real functions I – complex-analytic structure and integrable real functions. *Transnational Journal of Mathematical Analysis and Applications*, ***6***(1), 15–61.

[5] Kanwal R.P. (1998). *Function theory and technique* (2nd Edition), Birkhauser, Boston, pp. 1-17.

[6] Saichev A.I., and Woyczynski W.A. (1997). *Distributions in the physical and engineering sciences*, Boston, MA, Birkhauser, pp. 1-38.

[7] Khuri A.I. (2004). Applications of dirac's delta function in statistics. *International Journal of Mathematical Education in Science and Technology*, ***35***(2). 185-195.

[8] Brewster R.A., and Franson J.D. (2018.). Generalized delta functions and their use in quantum optics. *Journal of Mathematical Physics*, ***59***, 012102-17.

[9] Chakraborty S. (2008). Some Applications of Dirac-delta Function in statistics for more than one random variable. *Application and Applied Mathematics Journal* ***3***(1), 42-54.

[10] Onural L. (2006). Impulse functions over curves and surfaces and their application to diffraction. *Journal of Mathematical Analysis and Applications*, ***322***(1), 18-27.

[11] Sastry S.S. (2009). *Advanced engineering mathematics*. PHI learning ltd, New Delhi, pp. 36-57.

[12] Zhang L. (2018). On the calculus of dirac-delta function with some applications. *International Journal of Mathematics Trends and Technology*, ***56***(4),.258-270.

[13] Dimitriyadis I., and Oney, U.N. (2009). Deductibles in health insurance. *Journal. of*

Computational and Applied Mathematics, ***233(***1*)*, 51-60.

[14] Woodard J.D., and Yi. J. (2018). Estimation of insurance deductible demand under endogenous premium rates. *Journal of Risk and Insurance*. ***9999***, 1-24.

[15] Zhuang W. Z. C., and Hu T. (2008). Optimal allocation of policy limits and deductibles under distortion risk measures. *Insurance mathematics and Economics*, ***44***(3), 409-414.

[16] Bakar A.S., Hamzah N.A., Maghsoudi M., and Nadarajah S. (2015). Modelling loss data using composite models. *Insurance: Mathematics and Economics*. ***61***, 146-154.

[17] Pacakova V., and Brebera D. (2015). Loss distributions and simulations in general insurance and re-insurance. *International Journal of Mathematics and Computers in Simulation*. ***9***, 159-167.

[18] Thogersen J. (2016). Optimal premium as a function of the deductible: customer analysis and portfolio characteristics. *Risks*. ***4***, 1-19.

[19] Liu H., and Wang R. (2017). Collective risk models with dependence uncertainty. *ASTIN Bulletin*: *The Journal of the IAA*, ***47***(2), 361–389.

[20] Woodard J.D., and Yi J. (2020). Estimation of insurance deductible demand under endogenous premium rates. *Journal of Risk & Insurance, The American Risk and Insurance Association*. ***87***(2), 477-500.

[21] Ogungbenle, M.G. (2021b). On empirical analysis of Gompertzian mortality dynamics. *Journal of Science-FAS-SEUSL*, ***2***(1), 1-18.

[22] Hua L., and Cheung K.C. (2008). Worst allocation of policy limits and deductibles. *Insurance mathematics and Economics* ***40***(1), 93-98.

[23] Zacaj O., Dhamo E., and Shehu S. (2015). Modelling individual claims for motor third party liability of insurance companies in Albania. *International Journal of innovation in science and mathematics*, ***3***(3), 174-178.

[24] Sakthivel K.M., and Rajitha C.S. (2017). Artificial intelligence for estimation of future claim frequency in non- life insurance. *Global Journal of Pure and Applied Mathematics*. ***13***(6), 1701-1710.

[25] Actuarial Community (2018). *Loss data analytics*, arXiv:1808.06718v1 [q-fin.RM], 74-77.

[26] Garrido J. Genest C. and Schulz J. (2016). Generalized linear models for dependent frequency and severity of insurance claims. *Insurance: Mathematics and Economics*, ***70***, 205 – 215.

[27] Frees E. W. Lee G., and Yang (2016). Multivariate frequency-severity regression models in insurance. *Risks*, ***4***(1), 4.

[29] Park S. C. Kim J. H. and Ahn J. Y. (2018). Does hunger for bonuses drive the dependence between claim frequency and severity? *Insurance: Mathematics and Economics*, ***83***, 32–46.

[28] Valecky J. I. R. I. (2016). Modelling claim frequency in vehicle insurance. *Acta Universitatis Agriculturae et Silviculturae Mendelianae Brunensis*, 64(2), 683-689.

[30] Afify A.Z., Gemeay A.M., and Ibrahim N.A. (2020). The heavy tailed exponential distribution risk measures estimation and application to actuarial data. *Mathematics*, ***8***, 1276, 1-28.