 

RESEARCH ARTICLE

# THE MINIMUM NORMALIZED CUT VALUE OF TADPOLE GRAPHS

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## ABSTRACT

The Tadpole graph $T_{n,k}$ is a Lollipop type graph obtained by joining a one vertex of cycle graph $C_n$ to the end vertex of a path graph $P_k$. The normalized cut is a measure of disassociation between two groups which computes the cut cost as a fraction of the total edge connections to all the vertices in the graph. This research focuses on deriving a formula to find the minimum normalized cut value of Tadpole graphs.

**Keywords**: *Tadpole graph, Normalized cut, Lollipop graph*

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## 1. INTRODUCTION

A graph is a mathematical structure $G = (V, E)$ which consist of two finite sets namely $V$, the set of vertices (nodes or points) and $E$, the set of 2-element subsets of $V$, known as edges (arcs or lines). The Tadpole graph $T_{n,k}$ is similar to a Lollipop type graph, which is constructed by joining an end point of a path graph $P_k$ to one vertex of the cycle graph $C_n$ by a graph bridge. Tadpole graphs have been studied by several researchers for different aspects. Degree-based connectivity indices such as hyper Zagreb and Randić index of Tadpole graph has been studied by Kavitha *et.al* [4]. Laceability properties associated with Tadpole graphs have been studied in [5]. Equitable coloring of central, middle, total and line graphs of Tadpole graphs are discussed in [6]. Fibonacci number of the Tadpole graph through algebraic and combinatorics method was discussed in [7]. Domination number of Tadpole graph was studied in [8]. Sirinivasa *et.al* studied the characteristics of a typical Tadpole graph and its line graph by applying graph colouring and linear algebra methods [9]. In the real world, many practical applications of graph theory can be

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found in various fields. For an example, graph theory plays an important role in representing networks of communication, data organization, computational devices, the flow of computation, etc. In some circumstances, the computers in a computer network are connected in a structure like Tadpole graphs. So, failure of one or several structures will inactivate the whole system and the effect of removing a path from a network can be studied by removing edges from the graph. Therefore, the studying of minimum normalized cut value of Tadpole graphs is important. In 2013, Perera et.al derived a formular for the minimum normalized cut value of Lollipop graphs [1]. In the literature, there were no such formulas derived for the normalized cut of Tadpole graphs. The novelty of this research is deriving a formula for the minimum normalized cut of the Tadpole graphs by varying the path length and the cycle length.

A graph $G = (V,E)$ can be partitioned into two disjoint sets, $A$,$B$ such that $A \cup B = V$, and $A \cap B = \emptyset$, by simply removing the edges connecting the two parts. The degree of dissimilarity between these two components can be computed as total weight of the edges that have been removed, which is called a cut. The cut value is defined by

$$Cut(A,B) = \sum_{u\in A, v\in B} w(u,v),$$

where $w(u,v)$ denotes the weight of the edge $(u,v)$.

In 1993, Wu and Leahy introduced a new clustering method based on a minimum cut criterion [3]. To avoid the unnatural bias for partitioning out small sets of points in their method, in 2000, Shi and Malik proposed a novel global criterion, called the normalized cut (Ncut), for segmenting the graph assuming image segmentation as a graph partitioning problem [2]. The normalized cut criterion measures both the total dissimilarity between the different groups as well as the total similarity within the groups. It is defined as

$$Ncut(A,B) = \frac{Cut(A,B)}{assoc(A,V)} + \frac{Cut(A,B)}{assoc(B,V)},$$

where $assoc(A,V) = \sum_{u\in A, v\in V} w(u,v)$ is the total connections from vertices in $A$ to all other vertices in the graph and $assoc(B,V)$ can be similarly defined.

Throughout this paper, we assumed that graphs are unweighted and undirected. All the simulations required to satisfy our objectives were implemented by using the MATLAB software.

## 2. MATERIAL AND METHODS

Considering the following two cases, we will define the normalized cut value of a Tadpole graph.

**Case 1:** Consider a cut through the path as given in the Figure 1.

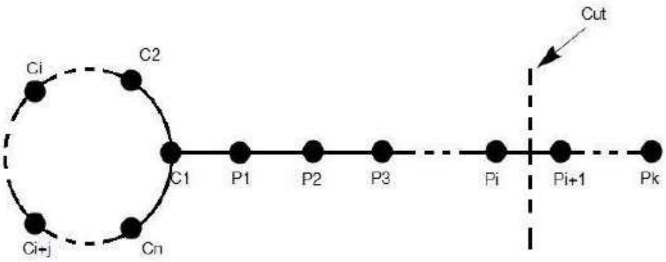


**Figure 1:** Tadpole graph in which the cut is done on the path graph

By using the definition of normalized cut value, we can write

$$Ncut = \frac{1}{k-I+1} + \frac{1}{n+I},$$

where $I \in \mathbb{Z}^{+}$ is the number of edges from the vertex $C_1$ to the cut edge of the path graph including the cut edge.

**Case II:** Consider a cut through the cycle as in the Figure 2.

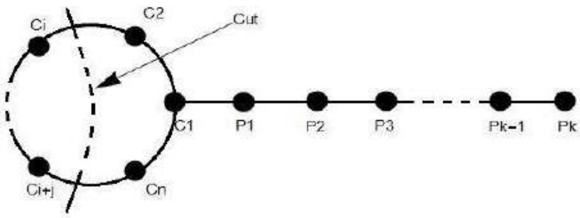


**Figure 2:** Tadpole graph in which the cut is done on the cycle graph

$$Ncut = \frac{2}{J+1} + \frac{2}{k+n-J+1},$$

where $J \in \mathbb{Z}^+$ is the number of vertices of the cycle graph in the left side of the cut as in the Figure 1.2. To determine the minimum value of normalized cut, we must identify the suitable cut point. We will derive the formula to identify the cut points of the graph and express it as Theorem 1 given below.

**Theorem 1:** Let $T_{n,k}$ be a Tadpole graph, where $n \geq 3$ and $k \geq 2$. Then the following formulae can be identified for the cut point $I$, cut is done on path and cut point $J$, the cut is done on cycle.

**Case I**: For $k \leq n + 2$

(i). $I = 1$ for $n < 10$

(ii). For $10 \leq n \leq 25$, when $k = \begin{cases} 2 & : 10 \leq n < 16 \\ 2,3 & : 16 \leq n < 22 \\ 2,3,4 & : 22 \leq n < 26 \end{cases}$

$$J = \begin{cases} \frac{n+k}{2} & : \quad n, k - both\ odd\ or\ even \\ \frac{n+k-1}{2} & : n - even\ and\ k - odd\ or\ n - odd\ and\ k - even, \end{cases}$$

For all $k$ values except the values mentioned above, $I = 1$.

**Case II**: $k > n + 2$

$$I = \begin{cases} \frac{k-n}{2} & : \quad n, k - \text{both are odd or both are even} \\ \frac{k-(n-1)}{2} & : \quad n - \text{even and } k - \text{odd or } n - \text{odd and } k - \text{even} \end{cases}$$

$$J = \frac{k+n}{2}$$

**Proof 1:** Let $f(i)$ be the Ncut value, when the cut is done on the path graph at $I = i$. Then

$$f(i) = \frac{1}{k-i+1} + \frac{1}{n+i}$$

Let $g(j)$ be the Ncut value, when the cut is done on the cycle graph at $J = j$. Then

$$g(j) = \frac{2}{j+1} + \frac{2}{k+n-j+1}$$

Consider $f(i) = \frac{1}{k-i+1} + \frac{1}{n+i}$. Stationary points can be obtained by $f'(i) = 0$.

Accordingly, $f'(i) = \frac{(n+i)^2-(k-i+1)^2}{(k-i+1)^2(n+i)^2} = 0$.

Since $\max(i) = k, \ i \neq k+1$, and $(k-i+1)^2(n+i)^2 > 0$. Therefore, $i = \frac{k-(n-1)}{2}$.

When $i < \frac{k-(n-1)}{2}$, we will consider $i = \frac{k-n}{2}$ $(i \in \mathbb{Z})$.

$$f'\left(\frac{k-n}{2}\right) = \frac{-16(n+k+1)}{(k+n+2)^2(n+k)^2} < 0\,, \ \ (k \geq 2, n \geq 3)$$

Now consider $i = \frac{k}{2}$ for $i > \frac{k-(n-1)}{2}$. Then we have

$$f'\left(\frac{k}{2}\right) = \frac{16(n+k+1)(n-1)}{(k+2)^2(2n+k)^2} > 0 \ \ (k \geq 2, n \geq 3)$$

Since $f'(i) < 0$ when $i < \frac{k-(n-1)}{2}$ and $f'(i) > 0$ when $i > \frac{k-(n-1)}{2}$, $f(i)$ is minimum at $i = \frac{k-(n-1)}{2}$ .

Therefore, when the cut is done on the path graph, minimum Ncut value can be obtained at $i = \frac{k-(n-1)}{2}$.

Similarly, we can find the stationary points of $g(j)$ by $g'(j) = 0$.

Accordingly, $g'(j) = \frac{2[(j+i)^2-(k+n-j+1)^2]}{(k+n-j+1)^2(j+1)^2} = 0$.

Since $\max(j) = n-1, \ j \neq k+n+1$, and $(k+n-j+1)^2(j+1)^2 > 0$. Therefore $j = \frac{k+n}{2}$.

When $j < \frac{k+n}{2}$, take $j = \frac{k}{2}$. $g'\left(\frac{k}{2}\right) = \frac{-32n(k+n+2)}{(k+2n+2)^2(k+2)^2} < 0 \ \ (n \geq 3, k \geq 2)$.

When $j > \frac{k+n}{2}$, take $j = \frac{k+n}{2} + 1$.

Then $g'\left(\frac{k+n}{2}+1\right) = \frac{8[(k+n+4)^2-(k+n)^2]}{(k+n+4)^2(k+n)^2} = \frac{64(k+n+2)}{(k+n+4)^2(k+n)^2} > 0 \ \ (n \geq 3, k \geq 2)$.

Since $g'(j) < 0$ when $j < \frac{k+n}{2}$ and $g'(j) > 0$ when $j > \frac{k+n}{2}$, $g(j)$ is minimum at $j = \frac{k+n}{2}$.

Therefore, when the cut is done on the cycle graph, minimum Ncut value can be obtained at $j = \frac{k+n}{2}$.

Next, we have to find $i$ and $j$ such that $i, j \in \mathbb{Z}$. $f(i)$ is minimum at $i = \frac{k-(n-1)}{2}$.

Since $n$ and $k$ may be odd or even, we need to consider the following two cases in order to approximate the value of $i$ to an integer.

**Case (a)***:* When $n$ is even and $k$ is odd or $n$ is odd and $k$ is even.

In this case, $i$ can select as, $i = \frac{k-(n-1)}{2} \in \mathbb{Z}$

**Case (b):** When both $n$ and $k$ are even or both are odd.

Then $i = \frac{k-(n-1)}{2} \notin \mathbb{Z}$.

Now, $i = \frac{k-n}{2}$ and $i = \frac{k-(n-2)}{2}$ are the nearest integers to $\frac{k-(n-1)}{2}$.

Since the value of $f(i)$ is same at $i = \frac{k-n}{2}$ and $i = \frac{k-(n-2)}{2}$. In this case, we select $i = \frac{k-n}{2}$. Similarly, since $g(j)$ is minimum at $j = \frac{k+n}{2}$, we will consider the following two cases to approximate the value of $j$ to the nearest integer.

**Case (c):** When $n$ is even and $k$ is odd or $n$ is odd and $k$ is even.

In this case, $j = \frac{n+k}{2} \notin \mathbb{Z}$.

Therefore, we consider $j = \frac{n+k-1}{2}$ and $j = \frac{n+k+1}{2}$, as they are the nearest integers to $\frac{n+k}{2}$.

Since the value of $g(j)$ is same as at $j = \frac{n+k-1}{2}$ and $j = \frac{n+k+1}{2}$, we will select $j = \frac{n+k-1}{2}$.

**Case (d):** When both $n$ and $k$ are even or both are odd, $j = \frac{n+k}{2} \in \mathbb{Z}$.

Now we can give the proof for the main theorem using the best cut points obtained above as mentioned in case(a) to case (d). We will revisit the formula derived for the minimum Ncut value, and give the proof for the main Theorem 1.

**Case I**: $k \leq n + 2$

**Sub-case (I): When both $n, k$ are odd or both are even**

$i = \frac{k-n}{2}$ (case (b))

$i \leq \frac{(n+2)-n}{2}$ ( $k \leq n+2$) implies $i = 1$

$j = \frac{n+k}{2}$ (case(d))

In this case, $min_i\{f(i)\} = f(1) = \frac{1}{k} + \frac{1}{n+1} = \frac{n+k+1}{k(n+1)}$, $min_i\{g(j)\} = \frac{2}{j+1} + \frac{2}{k+n-j+1} = \frac{8}{k+n+2}$.

The two values $min_i\{f(i)\}$ and $min_i\{g(j)\}$ are depending on $n$ and $k$. Therefore, we consider following special cases under case (i).

***Case (i)****:* $n \geq 10, k = 2$

$$f(1) = \frac{n+3}{2(n+1)}, \qquad min_j\{g(j)\} = \frac{8}{n+4}$$

$$f(1) - min_j\{g(j)\} = \frac{\left(n - \frac{9}{2}\right)^2 - \frac{97}{4}}{2(n+1)(n+4)} > 0, \qquad \forall n \geq 10$$

Therefore, $f(1) > min_j\{g(j)\}$. In this case minimum Ncut can be obtained when the cut is done at $j = \frac{n+k}{2}$, $(\forall n \geq 10, k = 2)$. For $\forall n < 10, k = 2$, the minimum Ncut is obtained when the cut is done at $i = 1$.

***Case (ii)****:* $n \geq 16, k = 3$

$$f(1) = \frac{n+4}{3(n+1)}, \qquad min_j\{g(j)\} = \frac{8}{n+5}$$

$$f(1) - min_j\{g(j)\} = \frac{\left(n - \frac{15}{2}\right)^2 - \frac{241}{4}}{3(n+1)(n+5)} > 0, \qquad \forall n \geq 16, k = 3$$

Therefore, $f(1) > min_j\{g(j)\}$. In this case the minimum Ncut can be obtained when the cut is done at $j = \frac{n+k}{2}$, $(\forall n \geq 16, k = 3)$. For $\forall n < 16, k = 3$, the minimum Ncut is obtained when the cut is done at $i = 1$.

***Case (iii)****:* $n \geq 22, k = 4$

$$f(1) = \frac{n+5}{4(n+1)}, \qquad min_j\{g(j)\} = \frac{8}{n+6}$$

$$f(1) - min_j\{g(j)\} = \frac{\left(n - \frac{21}{2}\right)^2 - \frac{449}{4}}{4(n+1)(n+6)} > 0, \quad \forall n \geq 22, k = 4$$

Therefore, $f(1) > min_j\{g(j)\}$. In this case, the minimum Ncut can be obtained when the cut is done at $j = \frac{n+k}{2}$, $(\forall n \geq 22, k = 4)$. For $\forall n < 22, k = 4$, the minimum Ncut is obtained when the cut is done at $i = 1$.

***Case (iv):*** $n \geq 27$

For this case, we obtain the sequence of $n$ values corresponding to the different $k$ values as: 27, 33, 39, 45, 51, 57, 62, 68, 74, 80,…. Using a computer simulation, we will summarize the $n$ and $k$ values used to obtain the minimum normalized cut as given in the Table 1. According to the results of the proof, for a given $n$ and $k$ value, the minimum normalized cut is equal to $f(1)$ or $minj\{g(j)\}$.

| | Minimum Ncut Value | |
|---|---|---|
| $k$–value | $min_j\{g(j)\}$ | $f(1)$ |
| 2 | $n \geq 10$ | $n < 10$ |
| 3 | $n \geq 16$ | $n < 16$ |
| 4 | $n \geq 22$ | $n < 22$ |
| 5 | $n \geq 27$ | $n < 27$ |
| 6 | $n \geq 33$ | $n < 33$ |
| 7 | $n \geq 39$ | $n < 39$ |
| … | … | … |
| … | … | … |
| 25 | $n \geq 144$ | $n < 144$ |
| 26 | $n \geq 150$ | $n < 150$ |
| 27 | $n \geq 156$ | $n < 156$ |

**Table 1:** Criteria to achieve minimum Ncut value of Tadpole graphs for $k \leq n + 2$

***Subcase (II): When n is even and k is odd or n is odd and k is even***

In this case, the proof is similar when both $n$ and $k$ are odd or both are even (sub-case(I)).

***Case (II):*** $k > n + 2$

$$i = \frac{k-(n-1)}{2}$$

$$i > \frac{(n+2)-(n-1)}{2} \quad \text{, since } k > n+2$$

$$i > \frac{3}{2} \Rightarrow i \geq 2 \quad (i \in \mathbb{Z}^{+})$$

***Subcase (I): When n is even and k is odd or n is odd and k is even***

$$i = \frac{k-(n-1)}{2}, \qquad j = \frac{n+k-1}{2} \in \mathbb{Z}.$$

$$min_i\{f(i)\} - min_j\{g(j)\} = \frac{1}{k-i+1} + \frac{1}{n+i} - \frac{2}{j+1} - \frac{2}{k+n-j+1}$$

$$= \frac{-4}{k+n+3} < 0; \quad (n \geq 3, k \geq 5)$$

Therefore $min_i\{f(i)\} < min_j\{g(j)\}$

Hence, the Ncut value is minimum, when the cut is done at $i = \frac{k-(n-1)}{2}$, $\forall k > n+2$.

***Subcase (II): When both n and k are even or both are odd.***

$$i = \frac{k-n}{2}, \quad j = \frac{n+k}{2}$$

$$min_i\{f(i)\} - min_j\{g(j)\} = \frac{1}{k-i+1} + \frac{1}{n+i} - \frac{2}{j+1} - \frac{2}{k+n-j+1}$$

$$= \frac{-4(k+n-1)}{(k+n)(k+n+2)} < 0 \;\; ; (n \geq 3, k \geq 5)$$

Therefore, $min_i\{f(i)\} < min_j\{g(j)\}$.

Hence, the Ncut value is minimum when the cut is done at $i = \dfrac{k-n}{2}$.

Then the proof of the formulae which were derived to obtain the place where the cut should be done on the Tadpole graph in order to obtain the minimum normalized cut value is completed. Using the normalized cut value (Ncut) equations, in (1) and (2), the exact Ncut values can be obtained.

## 3. RESULTS AND DISCUSSION

In this study, we have derived formulae given in the Theorem 1 for computing the minimum normalized cut value of Tadpole graphs with $n \leq 25$ satisfying the condition $n \leq k + 2$. Some of the results have been verified from the simulations as listed in the Table 1. It is remained for the future to study the minimum Ncut value of Tadpole graphs with $n > 25$ with the condition $n \leq k + 2$.

## CONCLUSION

This research focused on finding the minimum normalized cut value of Tadpole graphs. According to the literature survey, there were no formulae for minimum normalized cut value of Tadpole graphs. Considering several cases and simulation results, we have derived formulae for finding the cut points which caused to minimize the normalized cut value of all Tadpole graphs with $k > n + 2$ and the Tadpole graphs with $k \leq n + 2$ and $n \leq 25$, where $n$ is the number of vertices of the cycle graph and $k$ is the number of vertices of the path graph.

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