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RESEARCH ARTICLE

WEIGHTED MIXED TWO PARAMETER ESTIMATOR IN MULTIPLE LINEAR REGRESSION MODEL IN THE PRESENCE OF MULTICOLLINEARITY

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ABSTRACT

In this article, we introduce a new estimator, namely Weighted Mixed Two Parameter Estimator (WMTPE) to estimate the regression coefficients when the multicollinearity is present, and the stochastic restrictions are available in addition to the sample model. The proposed estimator is compared with some biased estimators in the Mean Square Error Matrix (MSEM) sense. To illustrate the theoretical findings, a Monte Carlo simulation study is conducted and a numerical example is used. From the theoretical and numerical results, it could be concluded that the proposed estimator performs better than other existing estimators used in this study.

Keywords: *Multicollinearity, Mixed Estimator, Biased Estimator, Weighted Mixed Two Parameter Estimator, Mean Square Error Matrix*

1. INTRODUCTION

In place of the Ordinary Least Square Estimator (OLSE), biased estimators have been proposed in the regression analysis in order to mitigate the multicollinearity issue. Firstly, Hoerl and Kennard [1] proposed Ridge Estimator (RE). Followed by Hoerl and Kennard [1], Liu Estimator (LE) (Liu [2]) and Almost Unbiased Two Parameter Estimator (AUTPE) (Wu and Yang [3]) have been proposed. An alternative method to deal with multicollinearity problem is to consider parameter estimation with some

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restrictions on the unknown parameters, which may be exact, or stochastic. The existence of stochastic prior information, in addition to the sample model leads to the principle of the Mixed Estimator (ME) introduced by Theil and Goldberger [4]. Later, Schaffrin and Toutenburg [5] proposed the Weighted Mixed Estimator (WME).

Then, Hubert and Wijekoon [6] and Li and Yang [7] proposed the Stochastic Restricted Liu Estimator (SRLE) and Stochastic Mixed Ridge Estimator (SMRE) respectively. In this article, we propose another biased estimator, namely Weighted Mixed Two Parameter Estimator (WMTPE) to estimate the regression coefficients when multicollinearity is present. In the literature, researchers seek the best estimator for multicollinearity, using sample or combined sample and prior information. Recognizing that combining two distinct estimators could potentially leverage the strengths of each, we are motivated to propose a novel estimator by amalgamating AUTPE and WME. Furthermore, the proposed estimator is compared with some existing biased estimators through both numerical and theoretical analyses.

The organization of the paper is as follows. The model specification and estimation were given in section 2. We proposed the estimator and obtained its stochastic properties in section 3. In section 4, the MSE matrix of the proposed estimator was compared with some biased estimators. A Monte Carlo simulation study was done and a numerical example was used to illustrate the theoretical findings in section 5. Finally, some concluding remarks were given in section 6.

2. MODEL SPECIFICATION AND ESTIMATION

To describe the problem, we consider the standard multiple linear model

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I) \tag{2.1}$$

where y is an $n\times 1$ vector of observations on the response variable, X is an $n\times p$ full column rank matrix of observations on p non-stochastic explanatory regressors variables, β is a $p\times 1$ vector of unknown parameters associated with p regressors and ε is an $n\times 1$ vector of disturbances with $E(\varepsilon) = \underset{\sim}{0}$ and the dispersion matrix $D(\varepsilon) = \sigma^2 I$.

In addition to sample model (2.1), let us be given some prior information about β in the form of a set of q independent stochastic linear restrictions as follows:

$$r = R\beta + \delta + \upsilon,\ \upsilon \sim N\left(0, \sigma^2\Omega\right), \tag{2.2}$$

where r is an $q \times 1$ stochastic known vector R is a $q \times p$ of full row rank $q \le p$ with known elements, δ is non-zero $q \times 1$ unknown vector and υ is an $q \times 1$ random vector of disturbances and Ω is assumed to be known and positive definite. Further, it is assumed that υ is stochastically independent of ε. i.e. $E\left(\varepsilon\upsilon'\right) = 0$. Note that when the stochastic restrictions are correct then $\delta = E\left(r\right) - R\beta = 0$. However, in practical situation, it is not possible to have correct stochastic restrictions. Therefore, we consider the biased stochastic restrictions (2.2) in this paper.

The Ordinary Least Square Estimator for the model (2.1) and Mixed Estimator (Theil and Goldberger [4]) due to a stochastic prior restriction (2.2) are given by

$$\hat{\beta}_{OLSE} = S^{-1}X'y \text{ and } \hat{\beta}_{ME} = \left(S + R'\Omega^{-1}R\right)^{-1}\left(X'y + R'\Omega^{-1}r\right) \tag{2.3}$$

respectively, where $S = X'X$.

Since $\left(S + R'\Omega^{-1}R\right)^{-1} = S^{-1} - S^{-1}R'\left(\Omega + RS^{-1}R'\right)^{-1}RS^{-1}$, the ME can be rewritten as

$$\hat{\beta}_{ME} = \hat{\beta}_{OLSE} + S^{-1}R'\left(\Omega + RS^{-1}R'\right)^{-1}\left(r - R\hat{\beta}_{OLSE}\right). \tag{2.4}$$

The bias vector and mean squares error of ME can be respectively derived as follows:

$$b_1 = B\left(\hat{\beta}_{ME}\right) = H\delta \text{ and } MSE\left(\hat{\beta}_{ME}\right) = \sigma^2 A + b_1 b_1' \tag{2.5}$$

where $H = S^{-1}R'\left(\Omega + RS^{-1}R'\right)^{-1}$, $\delta = E\left(r\right) - R\beta$ and $A = \left(S + R'\Omega^{-1}R\right)^{-1}$.

The Weighted Mixed Estimator (WME) proposed by Schaffrin and Toutenburg [5] is given by

$$\hat{\beta}_{WME}\left(w\right) = \left(S + wR'\Omega^{-1}R\right)^{-1}\left(X'y + wR'\Omega^{-1}r\right);\ 0 \le w \le 1. \tag{2.6}$$

Note that

$$\left(S + wR'\Omega^{-1}R\right)^{-1} = S^{-1} - wS^{-1}R'\left(\Omega + wRS^{-1}R'\right)^{-1}RS^{-1}.$$

Therefore, the WME can be rewritten as

$$\hat{\beta}_{WME}\left(w\right) = \hat{\beta}_{OLSE} + wS^{-1}R'\left(\Omega + wRS^{-1}R'\right)^{-1}\left(r - R\hat{\beta}_{OLSE}\right).$$

The bias vector and MSE matrix of $\hat{\beta}_{WME}(w)$ can be obtained as

$$b_2 = B\left[\hat{\beta}_{WME}(w)\right] = wH_w\delta$$

and

$$MSE\left[\hat{\beta}_{WME}(w)\right] = \sigma^2\tilde{A}\left(S + w^2R'\Omega^{-1}R\right)\tilde{A} + b_2b_2', \quad (2.7)$$

respectively, where $\tilde{A} = \left(S + wR'\Omega^{-1}R\right)^{-1}$ and $H_w = S^{-1}R'\left(\Omega + wRS^{-1}R'\right)^{-1}$.

The Almost Unbiased Two Parameter Estimator (AUTPE) (Wu and Yang [3]) is given by

$$\hat{\beta}_{AUTPE}(k,d) = T(k,d)\hat{\beta}_{OLSE}, \quad (2.8)$$

where $T(k,d) = \left[I - k^2(1-d)^2(S+kI)^{-2}\right]$ for $k \geq 0$, $0 \leq d \leq 1$.

The bias vector and MSE matrix of AUTPE can be obtained as

$$b_3 = B\left[\hat{\beta}_{AUTPE}(k,d)\right] = -k^2(1-d)^2(S+kI)^{-2}\beta$$

and

$$MSE\left[\hat{\beta}_{AUTPE}(k,d)\right] = \sigma^2T(k,d)S^{-1}T(k,d) + b_3b_3' \quad (2.9)$$

respectively.

The Stochastic Restricted Liu Estimator (SRLE) proposed by Hubert and Wijekoon [6] is given by

$$\hat{\beta}_{SRLE}(d) = F_d\hat{\beta}_{ME}, \quad (2.10)$$

where $F_d = (S+I)^{-1}(S+dI)$.

The bias vector and MSE matrix of SRLE are given by

$$b_4 = B\left(\hat{\beta}_{SRLE}(d)\right) = (d-1)(S+I)^{-1}\beta + F_dH\delta$$

and

$$MSE\left(\hat{\beta}_{SRLE}(d)\right) = \sigma^2F_dAF_d' + b_4b_4' \quad (2.11)$$

respectively.

The Stochastic Mixed Ridge Estimator (SMRE) proposed by Li and Yang [7] is given by

$$\hat{\beta}_{SMRE}(d) = W_k\hat{\beta}_{ME}, \quad (2.12)$$

where $W_k = \left(I + kS^{-1}\right)^{-1}$.

The bias vector and MSE matrix of SMRE are given by

$$b_5 = B\left[\hat{\beta}_{SMRE}(k)\right] = -k\left(S + kI\right)^{-1}\beta + W_k H\delta$$

and

$$MSE\left[\hat{\beta}_{SMRE}(k)\right] = \sigma^2 W_k A W_k' + b_5 b_5' \tag{2.13}$$

respectively, where $b_5 = -k\left(S + kI\right)^{-1}\beta + W_k H\delta$.

Following Hubert and Wijekoon [6] and Li and Yang [7], we may write the Stochastic Restricted Almost Unbiased Two Parameter Estimator (SRAUTPE) as follows:

$$\hat{\beta}_{SRAUTPE}(k,d) = T(k,d)\hat{\beta}_{ME}. \tag{2.14}$$

The bias vector and MSEM of SRAUTPE are given by

$$b_6 = B\left[\hat{\beta}_{SRAUTPE}(k,d)\right] = -k^2(1-d)^2\left(S + kI\right)^{-2}\beta + T(k,d)H\delta$$

and

$$MSE\left[\hat{\beta}_{SRAUTPE}(k,d)\right] = \sigma^2 T(k,d)AT(k,d) + b_6 b_6' \tag{2.15}$$

respectively, where $b_6 = -k^2(1-d)^2\left(S + kI\right)^{-2}\beta + T(k,d)H\delta$.

3. THE PROPOSED ESTIMATOR

In this research, we propose another biased estimator, namely Weighted Mixed Two Parameter Estimator (WMTPE) by replacing WME in the place of OLSE in AUTPE as follows:

$$\hat{\beta}_{WMTPE}(k,d,w) = T(k,d)\hat{\beta}_{WME}(w). \tag{3.1}$$

Now, we will see the some properties of $\hat{\beta}_{WMTPE}(k,d,w)$.

Note that

1. When $k = 0$ then $\hat{\beta}_{WMTPE}(0,d,w) = \hat{\beta}_{WME}(w)$.
2. When $d = 1$ then $\hat{\beta}_{WMTPE}(k,1,w) = \hat{\beta}_{WME}(w)$.
3. When $k = 1$ then $\hat{\beta}_{WMTPE}(1,d,w) = \left[I - (1-d)^2\left(S + I\right)^{-2}\right]\hat{\beta}_{WME}(w)$, this is called as Stochastic Weighted Mixed Almost Unbiased Liu Estimator (SWMAULE) proposed by Liu et.al. [8].

4. When $d=0$ then $\hat{\beta}_{WMTPE}(k,0,w)=\left[I-k^2(S+kI)^{-2}\right]\hat{\beta}_{WME}(w)$, this is called as Stochastic Weighted Mixed Almost Unbiased Ridge Estimator (SWMAURE) proposed by Liu et.al. [8].
5. When $w=1$ then $\hat{\beta}_{WMTPE}(k,d,1)=T(k,d)\hat{\beta}_{ME}$.
6. When $w=1$ and $d=1$ then $\hat{\beta}_{WMTPE}(k,1,1)=\hat{\beta}_{ME}$.

Hence, the proposed estimator is a generalization of ME, WME, SWMAULE, SWMAURE and SRAUTPE.

We obtain the mean vector, bias vector, dispersion matrix and mean squares matrix of $\hat{\beta}_{WMTPE}(k,d,w)$ as follows:

$$E\left[\hat{\beta}_{WMTPE}(k,d,w)\right]=T(k,d)\beta+wT(k,d)H\delta, \tag{3.2}$$

$$B\left[\hat{\beta}_{WMTPE}(k,d,w)\right]=-k^2(1-d)^2(S+kI)^{-2}\beta+wT(k,d)H\delta, \tag{3.3}$$

$$D\left[\hat{\beta}_{WMTPE}(k,d,w)\right]=\sigma^2T(k,d)\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}T(k,d) \tag{3.4}$$

and

$$MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right]=\sigma^2T(k,d)\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}T(k,d)+b_7b_7' \tag{3.5}$$

respectively, where $b_7=-k^2(1-d)^2(S+kI)^{-2}\beta+wT(k,d)H\delta$.

4. MEAN SQUARES ERROR MATRIX COMPARISONS

When different estimators are available for the same parameter vector β in the linear regression model one, must solve the problem of their comparison.

For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$ under the MSEM criterion if and only if

$$M\left(\hat{\beta}_1,\hat{\beta}_2\right)=MSE\left(\hat{\beta}_1,\beta\right)-MSE\left(\hat{\beta}_2,\beta\right)\geq 0. \tag{4.1}$$

(see Hubert and Wijekoon [6] and Li and Yang [7])

4.1. Comparison between $\hat{\beta}_{AUTPE}(k,d)$ and $\hat{\beta}_{WMTPE}(k,d,w)$

We consider the MSE matrix difference between $\hat{\beta}_{AUTPE}(k,d)$ and $\hat{\beta}_{WMTPE}(k,d,w)$ as

$$MSE\left[\hat{\beta}_{AUTPE}(k,d)\right]-MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right]=D_1+b_3b_3'-b_7b_7', \tag{4.2}$$

where D_1 is the dispersion matrix difference between $\hat{\beta}_{AUTPE}(k,d)$ and $\hat{\beta}_{WMTPE}(k,d,w)$.

Therefore, $D_1 = \sigma^2 T(k,d) S^{-1} T(k,d) - \sigma^2 T(k,d) \tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A} T(k,d)$. Also, the matrix D_1 can be simplified as

$D_1 = \sigma^2 T(k,d)\left(S^{-1} - \tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A}\right) T(k,d) = \sigma^2 T(k,d)\left(S^{-1} - D_w\right) T(k,d)$

where $D_w = \tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A}$.

Now, we can consider

$$\begin{aligned} S^{-1} - D_w &= S^{-1} - \left(S + wR'\Omega^{-1}R\right)^{-1}\left(S + w^2 R'\Omega^{-1}R\right)\left(S + wR'\Omega^{-1}R\right)^{-1} \\ &= S^{-1} - \left(S + wR'\Omega^{-1}R\right)^{-1} + w(1-w) R'\Omega^{-1}R\left(S + wR'\Omega^{-1}R\right)^{-1} \end{aligned}$$

Now, applying lemma 3 given in the appendix, one can prove that

$$S^{-1} - \left(S + wR'\Omega^{-1}R\right)^{-1} = wS^{-1}R'\left(\Omega + wRS^{-1}R'\right)^{-1} RS^{-1}.$$

Therefore,

$S^{-1} - D_w = wS^{-1}R'\left(\Omega + wRS^{-1}R'\right)^{-1} RS^{-1} + w(1-w) R'\Omega^{-1}R\left(S + wR'\Omega^{-1}R\right)^{-1}$. Now, it can be said that the matrix $S^{-1} - D_w$ is a positive definite since

$wS^{-1}R'\left(\Omega + wRS^{-1}R'\right)^{-1} RS^{-1} > 0$ and $w(1-w) R'\Omega^{-1}R\left(S + wR'\Omega^{-1}R\right)^{-1} > 0$.

Hence $D_1 > 0$ since $T(k,d) > 0$.

Now according to lemma 2, $MSE\left[\hat{\beta}_{AUTPE}(k,d)\right] - MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right] \geq 0$ if and only if $b_7'\left(D_1 + b_3 b_3'\right)^{-1} b_7 \leq 1$. Now, the following theorem can be stated.

Theorem 4.1:

The estimator $\hat{\beta}_{WMTPE}(k,d,w)$ is superior to $\hat{\beta}_{AUTPE}(k,d)$ in the MSE matrix sense if and only if $b_7'\left(D_1 + b_3 b_3'\right)^{-1} b_7 \leq 1$.

4.2. Comparison between $\hat{\beta}_{WME}(w)$ and $\hat{\beta}_{WMTPE}(k,d,w)$

We consider the following MSE matrix difference

$$MSE\left[\hat{\beta}_{WME}(w)\right] - MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right] = D_2 + b_2 b_2' - b_7 b_7', \tag{4.3}$$

where $D_2 = \sigma^2 D_w - \sigma^2 T(k,d) D_w T(k,d)$.

Now, we consider

$$\sigma^2 D_w - \sigma^2 T(k,d) D_w T(k,d)$$
$$= \sigma^2 D_w - \sigma^2 \left[I - k^2 (1-d)^2 (S+kI)^{-2} \right] D_w \left[I - k^2 (1-d)^2 (S+kI)^{-2} \right]$$
$$= \sigma^2 k^2 (1-d)^2 (S+kI)^{-2} \left[(S+kI)^2 D_w + D_w (S+kI)^2 - k^2 (1-d)^2 D_w \right] (S+kI)^{-2}$$
$$= \sigma^2 k^2 (1-d)^2 (S+kI)^{-2} (L-M)(S+kI)^{-2}$$

where $N = (S+kI)^2 D_w + D_w (S+kI)^2$ and $M = k^2 (1-d)^2 D_w$.

We can notice that $N > 0$ and $M > 0$.

According to lemma 1 in the appendix, the matrix D_2 is a positive definite matrix if and only if $\lambda_{\max}(MN^{-1}) < 1$. Now applying lemma 2 in the appendix, we can prove that

$$MSE\left[\hat{\beta}_{WME}(w)\right] - MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right] \geq 0 \text{ if and only if } b_7'(D_4 + b_2 b_2')^{-1} b_7 \leq 1.$$

Now, the following theorem can be stated.

Theorem 4.2:

When $\lambda_{\max}(MN^{-1}) < 1$, the estimator $\hat{\beta}_{WMTPE}(k,d,w)$ is superior to $\hat{\beta}_{WME}(w)$ in the MSEM sense if and only if $b_7'(D_4 + b_2 b_2')^{-1} b_7 \leq 1$, where $\lambda_{\max}(MN^{-1})$ is the maximum eigenvalue of the matrix MN^{-1}.

4.3. Comparison between $\hat{\beta}_{SRAUTPE}(k,d)$ and $\hat{\beta}_{WMTPE}(k,d,w)$

We consider the following MSE matrix difference as

$$MSE\left[\hat{\beta}_{SRAUTPE}(k,d)\right] - MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right] = D_3 + b_6 b_6' - b_7 b_7',$$

where

$$D_3 = \sigma^2 T(k,d) A T(k,d) - \sigma^2 T(k,d) \tilde{A}\left(S + w^2 R'\Omega^{-1} R\right) \tilde{A} T(k,d)$$
$$= \sigma^2 T(k,d)\left[A - \tilde{A}\left(S + w^2 R'\Omega^{-1} R\right)\tilde{A}\right] T(k,d).$$

We clearly know that $A > 0$ and $\tilde{A}\left(S + w^2 R'\Omega^{-1} R\right)\tilde{A} > 0$.

Now according to lemma 1 in the appendix, the D_3 is a positive definite matrix if and only if

$\lambda_{\max}\left\{\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}A^{-1}\right\}<1$. By applying lemma 2, we can say that

$$MSE\left[\hat{\beta}_{SRAUTPE}\left(k,d\right)\right]-MSE\left[\hat{\beta}_{WMTPE}\left(k,d,w\right)\right]\geq 0$$

if and only if $b_7'\left(D_4+b_6b_6'\right)^{-1}b_7\leq 1$. Now, the following theorem can be stated.

Theorem 4.3:

When $\lambda_{\max}\left\{\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}A^{-1}\right\}<1$, the estimator $\hat{\beta}_{WMTPE}\left(k,d,w\right)$ is superior to $\hat{\beta}_{SRAUTPE}\left(k,d\right)$ in the MSE matrix sense if and only if $b_7'\left(D_4+b_6b_6'\right)^{-1}b_7\leq 1$.

4.4. Comparison between $\hat{\beta}_{SRLE}\left(d\right)$ and $\hat{\beta}_{WMTPE}\left(k,d,w\right)$

We consider the following MSE matrix difference as

$$MSE\left[\hat{\beta}_{SRLE}\left(d\right)\right]-MSE\left[\hat{\beta}_{WMTPE}\left(k,d,w\right)\right]=D_4+b_4b_4'-b_7b_7'$$

where $D_4=\sigma^2F_dAF_d'-\sigma^2T\left(k,d\right)\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}T\left(k,d\right)$.

We know that $F_dAF_d'>0$ and $T\left(k,d\right)\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}T\left(k,d\right)>0$

Therefore, the matrix D_4 is a positive definite matrix if and only if

$$\lambda_{\max}\left\{\left[T\left(k,d\right)\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}T\left(k,d\right)\right]\left(F_dAF_d'\right)^{-1}\right\}<1.$$

Now according to lemma 2 in the appendix, we can conclude that

$$MSE\left[\hat{\beta}_{SRLE}\left(d\right)\right]-MSE\left[\hat{\beta}_{WMTPE}\left(k,d,w\right)\right]\geq 0$$

if and only if $b_7'\left(D_4+b_4b_4'\right)^{-1}b_7\leq 1$. Now, we can state the following theorem.

Theorem 4.4:

When $\lambda_{\max}\left\{\left[T\left(k,d\right)\tilde{A}\left(S+w^2R'\Omega^{-1}R\right)\tilde{A}T\left(k,d\right)\right]\left(F_dAF_d'\right)^{-1}\right\}<1$, the estimator $\hat{\beta}_{WMTPE}\left(k,d,w\right)$ is superior to $\hat{\beta}_{SRLE}\left(d\right)$ in the MSE matrix sense if and only if $b_7'\left(D_4+b_4b_4'\right)^{-1}b_7\leq 1$.

4.5. Comparison between $\hat{\beta}_{SMRE}\left(k\right)$ and $\hat{\beta}_{WMTPE}\left(k,d,w\right)$

We consider the following MSE matrix difference as

$$MSE\left[\hat{\beta}_{SMRE}\left(k\right)\right]-MSE\left[\hat{\beta}_{WMTPE}\left(k,d,w\right)\right]=D_5+b_5b_5'-b_7b_7',$$

where $D_5 = \sigma^2 W_k A W_k - \sigma^2 T(k,d)\tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A}T(k,d)$.

We clearly know that $W_k A W_k > 0$ and $T(k,d)\tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A}T(k,d) > 0$.

The matrix D_5 is a positive definite matrix if and only if

$$\lambda_{\max}\left\{\left[T(k,d)\tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A}T(k,d)\right]\left(W_k A W_k\right)^{-1}\right\} < 1.$$

Now according to lemma 2 in the appendix,

$$MSE\left[\hat{\beta}_{SMRE}(k)\right] - MSE\left[\hat{\beta}_{WMTPE}(k,d,w)\right] \geq 0$$

if and only if $b_7'\left(D_4 + b_5 b_5'\right)^{-1} b_7 \leq 1$. Now, the following theorem can be stated.

Theorem 4.5:

When $\lambda_{\max}\left\{\left[T(k,d)\tilde{A}\left(S + w^2 R'\Omega^{-1}R\right)\tilde{A}T(k,d)\right]\left(W_k A W_k\right)^{-1}\right\} < 1$, the estimator $\hat{\beta}_{WMTPE}(k,d,w)$ is superior to $\hat{\beta}_{SMRE}(k)$ in the MSE matrix sense if and only if $b_7'\left(D_4 + b_5 b_5'\right)^{-1} b_7 \leq 1$.

5. NUMERICAL ILLUSTRATION

5.1. Monte Carlo Simulation

In this section, to illustrate the performance of the proposed estimator, we carry out a Monte Carlo simulation study by considering two different levels of multicollinearity. Following McDonald and Galarneau [9], and Kibria [10], the explanatory variables can be generated as follows:

$$x_{ij} = \left(1-\gamma^2\right)^{1/2} z_{ij} + \gamma z_{i,p+1}, \; i = 1,2,...,n, \; j = 1,2,...,p,$$

where z_{ij} is an independent standard normal pseudo random number, and γ is specified so that the theoretical correlation between any two explanatory variables is given by γ^2. The regression coefficients $\beta_1, \beta_2, ..., \beta_p$, are selected as the normalized eigen vector corresponding to the largest eigen value of $X'X$ matrix (Newhouse and Oman, [11]).
A dependent variable is generated by using the equation.

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \varepsilon_i, \; i = 1,2,...,n,$$

where ε_i is a normal pseudo random number with mean zero and variance σ_i^2. We consider $n = 50, p = 4$ and $\sigma_i^2 = 1$ in the simulation. Moreover, two different sets of correlations are considered by selecting the values as $\gamma = 0.8$ and 0.9.

Consider the following stochastic restrictions

$$r = R\beta + \delta + \upsilon \text{ where } R = (1, -2, -2, -2)', \text{ and } \nu \sim N\left(0, \hat{\sigma}^2\right).$$

Table 1-Table 9 are obtained by using estimated Scalar Mean Square Error (SMSE) values obtained by using equations given in (2.5), (2.7), (2.9), (2.11), (2.13), (2.15) and (3.5).

Table 1: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.8$, d = 0.01 and w = 0.01

k	***ME***	***WME***	***AUTPE***	***SRLE***	***SMRE***	***SRAUTPE***	***WMTPE***
1	0.78394	0.11537	0.76681	0.8172	2.13382	0.83474	0.7673
0.9	0.78394	0.11537	0.60687	0.8172	2.11511	0.67254	0.60732
0.8	0.78394	0.11537	0.46613	0.8172	2.09257	0.52939	0.46652
0.7	0.78394	0.11537	0.34456	0.8172	2.06487	0.40525	0.34487
0.6	0.78394	0.11537	0.2421	0.8172	2.03002	0.30005	0.24231
0.5	0.78394	0.11537	0.15869	0.8172	1.98486	0.21376	0.15877
0.4	0.78394	0.11537	0.09425	0.8172	1.92408	0.14644	0.09412
0.3	0.78394	0.11537	0.04869	0.8172	1.83812	0.09859	0.04822
0.2	0.78394	0.11537	0.02216	0.8172	1.70792	0.07303	0.02095
0.1	0.78394	0.11537	0.01768	0.8172	1.48802	0.09045	0.01398
0	0.78394	0.11537	0.15976	0.8172	0.78394	0.78394	0.11536

Table 2: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.9$, d=0.01 and w=0.01

k	***ME***	***WME***	***AUTPE***	***SRLE***	***SMRE***	***SRAUTPE***	***WMTPE***
1	1.23319	0.20804	1.01789	1.0377	1.39063	1.01614	1.01772
0.9	1.23319	0.20804	0.80538	1.0377	1.37063	0.80432	0.80519
0.8	1.23319	0.20804	0.61844	1.0377	1.34673	0.61799	0.61824
0.7	1.23319	0.20804	0.45703	1.0377	1.31765	0.45705	0.4568
0.6	1.23319	0.20804	0.32103	1.0377	1.28151	0.3214	0.32076
0.5	1.23319	0.20804	0.21031	1.0377	1.23545	0.21089	0.20997
0.4	1.23319	0.20804	0.12462	1.0377	1.17495	0.12539	0.12417
0.3	1.23319	0.20804	0.06363	1.0377	1.09267	0.06492	0.06297
0.2	1.23319	0.20804	0.02685	1.0377	0.97705	0.03077	0.02571
0.1	1.23319	0.20804	0.01541	1.0377	0.81791	0.03639	0.01244
0	1.23319	0.20804	0.29569	1.0377	1.23319	1.23319	0.20803

Table 3: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.8$, d=0.01 and w=0.9

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	0.78394	0.11537	0.15678	0.70331	2.13382	0.7753	0.1133
0.9	0.78394	0.11537	0.15678	0.70331	2.11511	0.77468	0.11328
0.8	0.78394	0.11537	0.15678	0.70331	2.09257	0.77408	0.11327
0.7	0.78394	0.11537	0.15679	0.70331	2.06487	0.7735	0.11327
0.6	0.78394	0.11537	0.1568	0.70331	2.03002	0.77295	0.11327
0.5	0.78394	0.11537	0.15683	0.70331	1.98486	0.77245	0.11328
0.4	0.78394	0.11537	0.15687	0.70331	1.92408	0.77203	0.1133
0.3	0.78394	0.11537	0.15694	0.70331	1.83812	0.77176	0.11335
0.2	0.78394	0.11537	0.15709	0.70331	1.70792	0.77182	0.11345
0.1	0.78394	0.11537	0.15745	0.70331	1.48802	0.77298	0.1137
0	0.78394	0.11537	0.15976	0.70331	0.78394	0.78394	0.11536

Table 4: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.9$, d=0.01 and w=0.9

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	1.23319	0.20804	0.29003	0.94455	1.39063	1.20184	0.20401
0.9	1.23319	0.20804	0.29002	0.94455	1.37063	1.20259	0.204
0.8	1.23319	0.20804	0.29002	0.94455	1.34673	1.20337	0.204
0.7	1.23319	0.20804	0.29002	0.94455	1.31765	1.20416	0.20401
0.6	1.23319	0.20804	0.29003	0.94455	1.28151	1.20498	0.20402
0.5	1.23319	0.20804	0.29006	0.94455	1.23545	1.20585	0.20404
0.4	1.23319	0.20804	0.2901	0.94455	1.17495	1.20678	0.20408
0.3	1.23319	0.20804	0.29018	0.94455	1.09267	1.20785	0.20414
0.2	1.23319	0.20804	0.29033	0.94455	0.97705	1.20924	0.20425
0.1	1.23319	0.20804	0.29074	0.94455	0.81791	1.21174	0.20455
0	1.23319	0.20804	0.29569	0.94455	1.23319	1.23319	0.20803

Table 5: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.8$, d = 0.9 and w = 0.9

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	0.78394	0.6569	0.15678	0.70331	2.13382	0.7753	0.65007
0.9	0.78394	0.6569	0.15678	0.70331	2.11511	0.77468	0.6495
0.8	0.78394	0.6569	0.15678	0.70331	2.09257	0.77408	0.64896
0.7	0.78394	0.6569	0.15679	0.70331	2.06487	0.7735	0.64843
0.6	0.78394	0.6569	0.1568	0.70331	2.03002	0.77295	0.64792
0.5	0.78394	0.6569	0.15683	0.70331	1.98486	0.77245	0.64746
0.4	0.78394	0.6569	0.15687	0.70331	1.92408	0.77203	0.64707
0.3	0.78394	0.6569	0.15694	0.70331	1.83812	0.77176	0.6468
0.2	0.78394	0.6569	0.15709	0.70331	1.70792	0.77182	0.64682
0.1	0.78394	0.6569	0.15745	0.70331	1.48802	0.77298	0.64775
0	0.78394	0.6569	0.15976	0.70331	0.78394	0.78394	0.6569

Table 6: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.9$, *d*=0.9 and *w*=0.9

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	1.23319	1.03839	0.29003	0.94455	1.39063	1.20184	1.01158
0.9	1.23319	1.03839	0.29002	0.94455	1.37063	1.20259	1.01226
0.8	1.23319	1.03839	0.29002	0.94455	1.34673	1.20337	1.01295
0.7	1.23319	1.03839	0.29002	0.94455	1.31765	1.20416	1.01366
0.6	1.23319	1.03839	0.29003	0.94455	1.28151	1.20498	1.01439
0.5	1.23319	1.03839	0.29006	0.94455	1.23545	1.20585	1.01516
0.4	1.23319	1.03839	0.2901	0.94455	1.17495	1.20678	1.01599
0.3	1.23319	1.03839	0.29018	0.94455	1.09267	1.20785	1.01693
0.2	1.23319	1.03839	0.29033	0.94455	0.97705	1.20924	1.01814
0.1	1.23319	1.03839	0.29074	0.94455	0.81791	1.21174	1.02029
0	1.23319	1.03839	0.29569	0.94455	1.23319	1.23319	1.03839

Table 7: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.8$, *d*=0.01 and *w* = 0.3

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	0.78394	0.17547	0.76681	0.8172	2.13382	0.83474	0.78657
0.9	0.78394	0.17547	0.60687	0.8172	2.11511	0.67254	0.62586
0.8	0.78394	0.17547	0.46613	0.8172	2.09257	0.52939	0.48428
0.7	0.78394	0.17547	0.34456	0.8172	2.06487	0.40525	0.36177
0.6	0.78394	0.17547	0.2421	0.8172	2.03002	0.30005	0.25825
0.5	0.78394	0.17547	0.15869	0.8172	1.98486	0.21376	0.17362
0.4	0.78394	0.17547	0.09425	0.8172	1.92408	0.14644	0.10772
0.3	0.78394	0.17547	0.04869	0.8172	1.83812	0.09859	0.06038
0.2	0.78394	0.17547	0.02216	0.8172	1.70792	0.07303	0.0316
0.1	0.78394	0.17547	0.01768	0.8172	1.48802	0.09045	0.0245
0	0.78394	0.17547	0.15976	0.8172	0.78394	0.78394	0.17547

Table 8: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.9$, *d*=0.01 and *w*=0.3

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	1.23319	0.3002	1.01789	1.0377	1.39063	1.01614	1.01692
0.9	1.23319	0.3002	0.80538	1.0377	1.37063	0.80432	0.80456
0.8	1.23319	0.3002	0.61844	1.0377	1.34673	0.61799	0.61774
0.7	1.23319	0.3002	0.45703	1.0377	1.31765	0.45705	0.45637
0.6	1.23319	0.3002	0.32103	1.0377	1.28151	0.3214	0.32035
0.5	1.23319	0.3002	0.21031	1.0377	1.23545	0.21089	0.20949
0.4	1.23319	0.3002	0.12462	1.0377	1.17495	0.12539	0.12354
0.3	1.23319	0.3002	0.06363	1.0377	1.09267	0.06492	0.0621
0.2	1.23319	0.3002	0.02685	1.0377	0.97705	0.03077	0.02467
0.1	1.23319	0.3002	0.01541	1.0377	0.81791	0.03639	0.01262
0	1.23319	0.3002	0.29569	1.0377	1.23319	1.23319	0.3002

Table 9: Estimated SMSE values of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $\gamma = 0.8$, d = 0.01 and w = 0.6

k	***ME***	***WME***	***AUTPE***	***SRLE***	***SMRE***	***SRAUTPE***	***WMTPE***
1	0.78394	0.35601	0.76681	0.8172	2.13382	0.83474	0.80692
0.9	0.78394	0.35601	0.60687	0.8172	2.11511	0.67254	0.64553
0.8	0.78394	0.35601	0.46613	0.8172	2.09257	0.52939	0.50322
0.7	0.78394	0.35601	0.34456	0.8172	2.06487	0.40525	0.37994
0.6	0.78394	0.35601	0.2421	0.8172	2.03002	0.30005	0.27559
0.5	0.78394	0.35601	0.15869	0.8172	1.98486	0.21376	0.19008
0.4	0.78394	0.35601	0.09425	0.8172	1.92408	0.14644	0.1233
0.3	0.78394	0.35601	0.04869	0.8172	1.83812	0.09859	0.07522
0.2	0.78394	0.35601	0.02216	0.8172	1.70792	0.07303	0.04663
0.1	0.78394	0.35601	0.01768	0.8172	1.48802	0.09045	0.04574
0	0.78394	0.35601	0.15976	0.8172	0.78394	0.78394	0.35601

From Table 1 and Table 2, it can be noticed that the estimator WMTPE has the smallest SMSE than ME, AUTPE, SRLE, SMRE and SRUTPE. Nevertheless, the estimator WME has the smallest SMSE than WMTPE. Based on Table 3 and Table 4, it can be concluded that the estimator WMTPE has the smallest SMSE value than other estimators. The estimator WME also has the smallest SMSE values than ME, WME, AUTPE, SRLE, SMRE and SRUTPE. From Table 5 and Table 6, we may conclude that the estimator AUTPE has the smallest SMSE than other estimators. However, the proposed estimator has also the smallest SMSE than ME, WME, SMRE, SRLE and SRAUTPE. According to Table 7, we can say that the estimator WMTPE has the smallest SMSE than SRAUTPE and SMRE for all *k* values. The estimator WMTPE has the smallest SMSE than SRLE. When k is small, the estimator WMTPE has the smallest SMSE than WME. The WMTPE has the smallest SMSE than ME except *k* = 1. However AUTPE has the smallest SMSE than WMTPE. From Table 8, we can conclude that the estimator WMRTPE has the smallest SMSE than SRAUTPE when $k \leq 0.8$. The WMTPE has the smallest SMSE than SRLE, SMRE, AUTPE and ME. The estimator WMTPE has the smallest SMSE than WME and AUTPE for some *k* values. Based on Table 9, we can say that the estimator WMTPE has the smallest SMSE than SRAUTPE, SMRE and SRLE. The estimator WMTPE has the smallest SMSE than WME, ME and AUTPE for some k values. The estimator WMTPE has the smallest SMSE than ME except for *k* = 1. However, the estimator AUTPE has the smallest SMSE than WMTPE. From Table 10, we can conclude that the estimator WMTPE has the smallest SMSE than SRAUTPE except *k* =1. The estimator WMTPE has the smallest SMSE than SMRE, SRLE and ME. The estimator WMTPE has the smallest SMSE than WME for some *k* values.

5.2. Numerical Example

To further illustrate the behavior of the proposed estimator, we consider the data set on Total National Research and Development Expenditures as a Percent of Gross National product. This data was initially studied by Gruber [12], and later considered by Akdeniz and Erol [13], Li and Yang [14] and Alheety and Kibria [15]. The data set is given below:

$$X = \begin{pmatrix} 1.9 & 2.2 & 1.9 & 3.7 \\ 1.8 & 2.2 & 2.0 & 3.8 \\ 1.8 & 2.4 & 2.1 & 3.6 \\ 1.8 & 2.4 & 2.2 & 3.8 \\ 2.0 & 2.5 & 2.3 & 3.8 \\ 2.1 & 2.6 & 2.4 & 3.7 \\ 2.1 & 2.6 & 2.6 & 3.8 \\ 2.2 & 2.6 & 2.6 & 4.0 \\ 2.3 & 2.8 & 2.8 & 3.7 \\ 2.3 & 2.7 & 2.8 & 3.8 \end{pmatrix} \text{ and } y = \begin{pmatrix} 2.3 \\ 2.2 \\ 2.2 \\ 2.3 \\ 2.4 \\ 2.5 \\ 2.6 \\ 2.6 \\ 2.7 \\ 2.7 \end{pmatrix}.$$

The four column of the 10×4 matrix X comprise the data on x_1 , x_2 , x_3 and x_4 respectively, and y is the predictor variable. Note that the eigenvalues of S are $\lambda_1 = 302.9626$, $\lambda_2 = 0.7283$, $\lambda_3 = 0.0447$ and $\lambda_4 = 0.0345$ and, the condition number of X is approximately 8781.53, which indicates a high multicollinearity among explanatory variables. The OLSE is given by

$$\hat{\beta}_{OLSE} = S^{-1}X'y = \left(0.6455, 0.0896, 0.1436, 0.1526\right)'$$

with $MSE\left(\hat{\beta}_{OLSE}, \beta\right) = 0.0808$ and $\hat{\sigma}^2 = 0.0015$.

We consider the same stochastic prior information used in the simulation study.

Table 10-Table 14 display estimated SMSE values computed using equations given in (2.5), (2.7), (2.9), (2.11), (2.13), (2.15) and (3.5).

Table 10: Estimated SMSE of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $d = 0.01$ and $w = 0.01$.

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	0.28638	0.24054	0.22248	0.20334	0.02339	0.1799	0.18391
0.9	0.28638	0.24054	0.17846	0.20334	0.0238	0.13756	0.14136
0.8	0.28638	0.24054	0.13952	0.20334	0.02439	0.10051	0.10406
0.7	0.28638	0.24054	0.10567	0.20334	0.02522	0.06885	0.07210
0.6	0.28638	0.24054	0.07691	0.20334	0.02639	0.04276	0.04563
0.5	0.28638	0.24054	0.05328	0.20334	0.02806	0.02258	0.02493
0.4	0.28638	0.24054	0.03488	0.20334	0.03055	0.01907	0.01063
0.3	0.28638	0.24054	0.02204	0.20334	0.03447	0.00405	0.00927
0.2	0.28638	0.24054	0.01592	0.20334	0.04163	0.01312	0.01055
0.1	0.28638	0.24054	0.02217	0.20334	0.06077	0.06064	0.04996
0	0.28638	0.24054	0.08079	0.20334	0.28638	0.28638	0.24054

Table 11: Estimated SMSE of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $d = 0.01$ and $w = 0.9$.

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	0.28638	0.0452	0.22248	0.20334	0.02339	0.1799	0.22169
0.9	0.28638	0.0452	0.17846	0.20334	0.0238	0.13756	0.17762
0.8	0.28638	0.0452	0.13952	0.20334	0.02439	0.10051	0.13861
0.7	0.28638	0.0452	0.10567	0.20334	0.02522	0.06885	0.10466
0.6	0.28638	0.0452	0.07691	0.20334	0.02639	0.04276	0.07576
0.5	0.28638	0.0452	0.05328	0.20334	0.02806	0.02258	0.05189
0.4	0.28638	0.0452	0.03488	0.20334	0.03055	0.00907	0.03310
0.3	0.28638	0.0452	0.02204	0.20334	0.03447	0.02405	0.01950
0.2	0.28638	0.0452	0.01592	0.20334	0.04163	0.01312	0.01164
0.1	0.28638	0.0452	0.02217	0.20334	0.06077	0.06064	0.01237
0	0.28638	0.0452	0.08079	0.20334	0.28638	0.28638	0.04515

Table 12: Estimated SMSE of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $d = 0.9$ and $w = 0.9$.

k	*ME*	*WME*	*AUTPE*	*SRLE*	*SMRE*	*SRAUTPE*	*WMTPE*
1	0.28638	0.24054	0.07934	0.19498	0.02339	0.27662	0.23208
0.9	0.28638	0.24054	0.07935	0.19498	0.0238	0.27714	0.23254
0.8	0.28638	0.24054	0.07937	0.19498	0.02439	0.27767	0.23301
0.7	0.28638	0.24054	0.07938	0.19498	0.02522	0.27821	0.23349
0.6	0.28638	0.24054	0.0794	0.19498	0.02639	0.27877	0.23400
0.5	0.28638	0.24054	0.07944	0.19498	0.02806	0.27937	0.23452
0.4	0.28638	0.24054	0.07948	0.19498	0.03055	0.28002	0.23510
0.3	0.28638	0.24054	0.07956	0.19498	0.03447	0.28076	0.23575
0.2	0.28638	0.24054	0.07969	0.19498	0.04163	0.28171	0.23657
0.1	0.28638	0.24054	0.07997	0.19498	0.06077	0.28321	0.23786
0	0.28638	0.24054	0.08079	0.19498	0.28638	0.28638	0.24054

Table 13: Estimated SMSE of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $d = 0.01$ and $w = 0.6$.							
k	***ME***	***WME***	***AUTPE***	***SRLE***	***SMRE***	***SRAUTPE***	***WMTPE***
1	0.28638	0.13198	0.22248	0.20334	0.02339	0.1799	0.19623
0.9	0.28638	0.13198	0.17846	0.20334	0.0238	0.13756	0.15309
0.8	0.28638	0.13198	0.13952	0.20334	0.02439	0.10051	0.11512
0.7	0.28638	0.13198	0.10567	0.20334	0.02522	0.06885	0.08236
0.6	0.28638	0.13198	0.07691	0.20334	0.02639	0.04276	0.05487
0.5	0.28638	0.13198	0.05328	0.20334	0.02806	0.02258	0.03280
0.4	0.28638	0.13198	0.03488	0.20334	0.03055	0.00907	0.01648
0.3	0.28638	0.13198	0.02204	0.20334	0.03447	0.00405	0.00673
0.2	0.28638	0.13198	0.01592	0.20334	0.04163	0.01312	0.00610
0.1	0.28638	0.13198	0.02217	0.20334	0.06077	0.06064	0.02576
0	0.28638	0.13198	0.08079	0.20334	0.28638	0.28638	0.13198

Table 14: Estimated SMSE of ME, WME, SRLE, SMRE, AUTPE, SRAUTPE and WMTPE for $d = 0.01$ and $w = 0.3$.							
k	***ME***	***WME***	***AUTPE***	***SRLE***	***SMRE***	***SRAUTPE***	***WMTPE***
1	0.28638	0.06684	0.22248	0.20334	0.02339	0.1799	0.20897
0.9	0.28638	0.06684	0.17846	0.20334	0.0238	0.13756	0.16532
0.8	0.28638	0.06684	0.13952	0.20334	0.02439	0.10051	0.12678
0.7	0.28638	0.06684	0.10567	0.20334	0.02522	0.06885	0.09334
0.6	0.28638	0.06684	0.07691	0.20334	0.02639	0.04276	0.06504
0.5	0.28638	0.06684	0.05328	0.20334	0.02806	0.02258	0.04191
0.4	0.28638	0.06684	0.03488	0.20334	0.03055	0.00907	0.02408
0.3	0.28638	0.06684	0.02204	0.20334	0.03447	0.00405	0.01191
0.2	0.28638	0.06684	0.01592	0.20334	0.04163	0.01312	0.00655
0.1	0.28638	0.06684	0.02217	0.20334	0.06077	0.06064	0.01328
0	0.28638	0.06684	0.08079	0.20334	0.28638	0.28638	0.06684

From Table 10, we can say that the estimator WMTPE has smallest SMSE than the ME and WME. Also, the estimator WMTPE has smallest SMSE than the other estimators for some k values. The estimator WMTPE has smallest SMSE than the AUTPE when $k \geq 0.2$. The estimator WMTPE has smallest SMSE than the SRLE except $k = 0$. From Table 11, we can conclude that the estimator WMTPE has smallest SMSE than the SRLE except $k = 1$. Moreover, the estimator WMTPE has smallest SMSE than the other estimators for some k values. According to Table 12, it can be observed that the estimator WMTPE has smallest SMSE than the ME, WME and SRAUTPE. Nevertheless, the estimators AUTPE and SRLE have smallest SMSE than WMTPE.

From Table 13, it has been observed that the estimator WMTPE has smallest SMSE than the ME and SRLE. The estimator WMTPE has smallest SMSE than the AUTPE. Based on Table 14, we can notice that the estimator WMTPE has smallest SMSE than the WME for some k values. The estimator WMTPE has smallest SMSE than the ME and AUTPE.

There are three unknown parameters in the proposed estimators such as k, d and w. Following Alheety and et al. [16] and Kaçıranlar and et al. [17], we can theoretically obtain the optimum value of k, d or w by minimizing Scalar Mean Square Error (SMSE) values with respect to k, d or w. However, in this paper, we have not made any attempt to estimate them.

6. CONCLUSION

In this paper, we proposed another biased estimator, namely Weighted Mixed Two Parameter Estimator (WMTPE) to estimate the regression coefficients in the presence of multicollinearity. The proposed estimator was compared with some biased estimators in the Mean Square Error Matrix (MSEM) sense. Finally, to illustrate the theoretical results, we performed a Monte Carlo Simulation study by considering two different levels of multicollinearity and used a numerical example. Based on the theoretical results and numerical illustrations, it can be concluded that that our proposed estimator is useful in practice.

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APPENDIX

Lemma 1: (Wang et al. [18])

Let $n \times n$ matrices $M > 0$, $N > 0$ (or $N \geq 0$), then $M > N$ if and only if $\lambda_1\left(NM^{-1}\right) < 1$, where $\lambda_1\left(NM^{-1}\right)$ is the largest eigenvalue of the matrix NM^{-1}.

Lemma 2: (Trenkler and Toutenburg, [19])

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be two linear estimator of β. Suppose that $D = D\left(\hat{\beta}_1\right) - D\left(\hat{\beta}_2\right)$ is positive definite then $\Delta = MSE\left(\hat{\beta}_1\right) - MSE\left(\hat{\beta}_2\right)$ is nonnegative definite if and only if $b_2'\left(D + b_1 b_1'\right)^{-1} b_2 \leq 1$, where b_j denotes the bias vector of $\hat{\beta}_j$, $j = 1, 2$.

Lemma 3: (Rao [20], p.33)

Let A be an $n \times n$ matrix, B an $(n \times m)$ matrix, C an $(m \times m)$ matrix, and D an $(m \times n)$ matrix, where the necessary inverse exist. Then

$$\left(A + BCD\right)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1}.$$