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RESEARCH ARTICLE

COMPUTATION OF DEGREE-DEPENDENT TOPOLOGICAL INDICES FOR BANANA TREE GRAPHS

K.K.K.R. Perera

Department of Mathematics, Faculty of Science, University of Kelaniya, Dalugama, Kelaniya 11600, Sri Lanka

ABSTRACT

A banana tree graph $B(n, k)$ is obtained by connecting one leaf of each of n copies of a k-star graph to a single root vertex that is distinct from all the stars. Due to their branching structure, banana tree graphs resemble certain molecular frameworks and therefore provide a useful mathematical model for studying degree-based topological indices. Such indices, including the Randić index, Zagreb indices, geometric–arithmetic index, and atomic-bond connectivity index, play an important role in chemical graph theory by correlating molecular structure with physicochemical properties. Banana trees also serve as convenient examples for analyzing graph invariants such as diameter, eccentricity, and chromatic number. In this study, we derive closed-form expressions for several degree-based topological indices of banana tree graphs in terms of the parameters n and k. A comparative analysis is carried out to examine the sensitivity of these indices to changes in branching structure. The results highlight how variations in structural parameters influence molecular descriptors, thereby demonstrating the relevance of banana tree graphs in modeling branched molecular systems within chemical graph theory.

Keywords: Banana tree, Topological index, Graph invariants, Dendrimer

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*Corresponding author: *kkkrperera@kln.ac.lk*

1. INTRODUCTION

Banana trees resemble certain molecular structures such as hydrocarbons, making them useful for studying topological indices like the Wiener index, ABC index, Zagreb indices, etc. Highly branched macromolecules such as dendrimers also exhibit tree-like architectures. For example, Poly(amidoamine) dendrimer consists of a central core from which multiple branching units grow generation by generation [21]. The repetitive branching structure of dendrimers can be effectively modeled using tree-based graphs. Banana tree graphs, with a central root connected to several star components.

Therefore, investigating topological indices of banana trees contributes to understanding structural descriptors of branched polymer systems. Banana trees are also favourite counter examples for computing and comparing graph invariants like radius, diameter, chromatic number, etc. since they have both tree-like and star-like properties. Structure of Banana tree is important when studying communication networks or hierarchical systems**,** where the central hubs (stars) feeding into a main server (the root). Banana tree graphs were studied from different perspectives. The degree based topological indices of Banana tree graphs were studied in [7]. Sebastian J.K. et al. [19] have discussed the pendant number of banana tree graphs. Locating the chromatic number of Banana tree was discussed in [9]. In [13], the authors proved that all banana trees and union of any number of stars are integral sum graphs. In [11], it was shown that Banana trees corresponding to the family of stars are graceful. Ahmad et. al. [14] computed closed form of the M-polynomial for the line graph of the banana tree, and from the M-polynomial, they recovered some degree-based topological indices. In [5], graceful labeling, Harmonious labeling, Zumkeller labeling of banana trees were calculated. The Fibonacci sum labels of Banana tree were studied in [6]. Tunc et al. [3] investigated the results about closeness and residual closeness of Banana Trees. Murusan et al. [20] studied the L(3,2,1) labeling of the banana tree graphs. In [8], the dominating set and domination number of the Banana tree graphs were discussed. Novelia et al. [10] determined the exact values of the reflexive edge strength of banana tree graphs $B_{2,n}$ and $B_{3,n}$. The radio mean D-distance number of banana trees was studied in [12]. The Sombor indices are used for predicting the thermodynamic characteristics of substances. In order to describe the complexity of graphs, graph entropy was proposed. In [16] Sombar indices and Somba entropy for the banana tree graphs were computed. Metric Dimension has various applications in real life. In [17] edge metric dimension of banana tree graphs was discussed. If every vertex of $V - D$, where D is a subset of V is adjacent to at least one vertex of D, then D is said to be the dominating set in the graph. The domination contraction number of a graph $ct\gamma(G)$ is defined as the minimum number of edges that must be contracted to reduce the dominance number. Simanungkalit et al. [18] determined the pattern of dominance side contraction numbers in banana tree graphs. According to the literature, most of the banana tree graphs were discussed to study graph invariant properties and some were related to the topological indices.

In chemistry, molecular structures are often modelled using molecular graphs. A molecular graph is a simple graph such that its vertices correspond to the atoms and the edges to the bonds of the molecule. Many branched hydrocarbons contain tertiary carbon atoms, where one carbon atom is bonded to three other carbon atoms commonly referred to as tri-branched hydrocarbons. For an example Isobutane or 2-methylpropane is a chemical compound whose molecular graph contains a vertex of degree three connected to three pendant vertices. Banana tree graphs with three star components attached to a root vertex exhibit a similar branching pattern. Therefore, banana trees can serve as simplified mathematical models for studying structural descriptors of tri-branched hydrocarbon frameworks. A topological index is a numerical value associated with chemical structures which describes the correlation between chemical structure and various physical properties, such as chemical reactivity and biological activity. Degree based topological indices like Zagreb, Randic, ABC etc. are sensitive to number of branches of vertices, distribution of carbon atoms in molecules and it is evident from the literature that those indices are correlated with boiling point, stability, reactivity, surface area of molecules too. The objective of this study is to find the closed form formula for the topological indices of Banana tree graphs, which were not taken into consideration.

2. MATERIAL AND METHODS

2.1 Preliminaries

A graph is an ordered pair denoted by $G = (V, E)$, where $V\,(G)$ and $E(G)$, represent the vertex set and edge set respectively. Tree is an acyclic connected graph and extremely useful in chemistry since most organic compounds have tree-like structures.

Banana tree graph $B(n,k)$ is a graph obtained by connecting one leaf of each n copies of a k star graph with a single root vertex that is distinct from all the stars. Those banana tree graphs are called uniform graphs, and these are simple, symmetric, and can be easily used for computations. Uniform Banana trees are used in network design and communication systems, where one central hub connects to several subnetworks and each subnetwork serves multiple end users. Figure 1 represents a general structure of Banana tree with n copies of star graphs each with k vertices. Figure 2 shows some banana tree graphs for different n and m values.

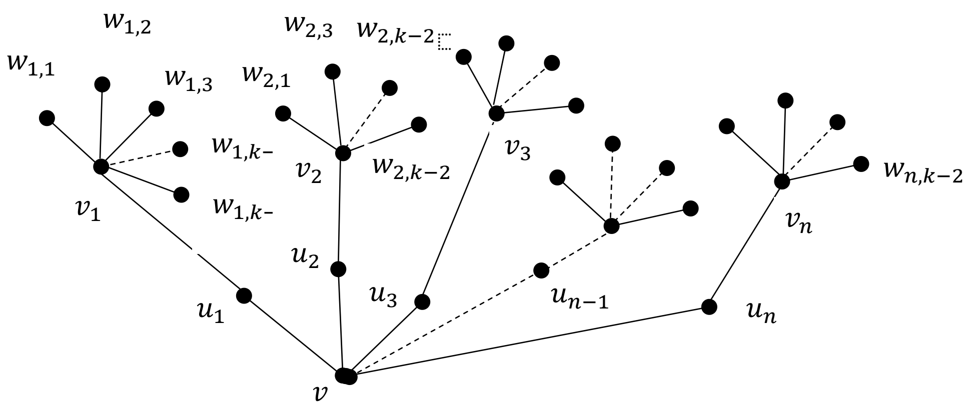
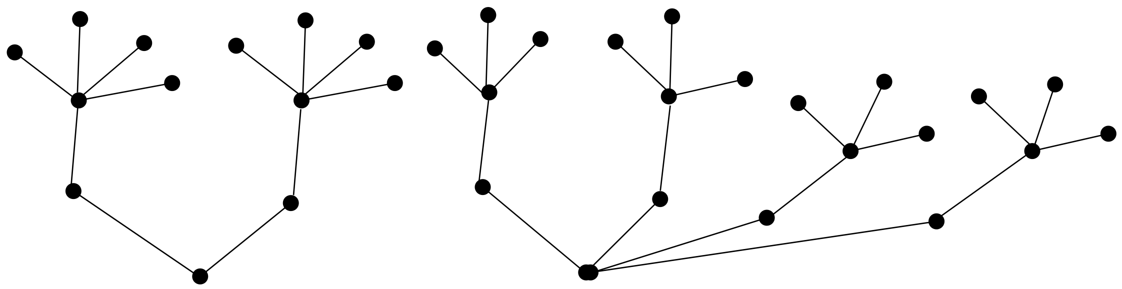


Figure 1: Banana Tree $B(n,k)$



(a) $B(2,6)$ (b) $B(4,5)$

Figure 2: Different Banana Trees

For two vertices x and y, the distance between x and y is denoted by $d_G(x,y)$ is the length of any shortest path connecting x and y. The eccentricity $\varepsilon_G(u)$ of a vertex u is defined as the largest distance between u and any other vertex v of G. i.e., $\varepsilon_G(u) = \max_{v \epsilon V(G)} d_G(u,v)$.

Gutman and Trinajstic [15] introduced first Zagreb index and second Zagreb index as

$M_1(G) = \sum_{v \in V(G)} (d_G(v))^2$ and $M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$.

The Zagreb indices are among the most fundamental degree-based topological descriptors in chemical graph theory. Since they depend exclusively on vertex degrees, they effectively quantify molecular branching, which plays a vital role in determining physicochemical and biological properties of organic compounds.

New version of Zagreb indices were introduced in [4] using eccentricity as follows:

$M_1^*(G) = \sum_{uv \in E(G)} \varepsilon_G(u) + \varepsilon_G(v)$, $M_1^{**}(G) = \sum_{v \in V(G)} (\varepsilon_G(v))^2$,

$$M_2^*(G) = \sum_{uv \in E(G)} \varepsilon_G(u) \varepsilon_G(v)$$

The following explains some of the topological indices used in this study:

Randić index of graph G is denoted by $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$.

Geometric arithmetic index [1] of G is denoted by $GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}$

Modified Zagreb index is defined as $M(G) = \sum_{uv \in E(G)} \frac{1}{d_u d_v}$.

Atomic-bond connectivity index is defined as $ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}$

For any $(u, v) \in E(G)$, let $S_u = \sum_{uv \in E(G)} d_v$. The fourth geometric connectivity index is defined as $ABC_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}$

The fifth geometric connectivity index [2] is $ABC_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S_u S_v}}{S_u + S_v}$

The Sanskruti index is defined as $S(G) = \sum_{uv \in E(G)} \left(\frac{S_u S_v}{S_u + S_v - 2}\right)^3$

All of the above indices are based on the degree of the vertices, and we will derive closed form expression for the topological indices of banana tree graphs using above mentioned topological indices.

3: RESULTS AND DISCUSSION

3.1 Computation of closed form formulas for Topological indices of Banana tree

Theorem 1: For a Banana tree $B(n,k)$, where $n \geq 2$ and $k \geq 3, n, k \in \mathbb{Z}^+$, we have the following:

$M_1^*(B(n,k)) = 11nk - 6n$

$M_1^{**}(B(n,k)) = 36nk - 31n + 9$

$M_2^*(B(n,k)) = 30kn - 28n$

Proof:

For a Banana tree $B(n,k)$, we can identify four types of vertices denoted by $v, u_i, v_i, w_{i,j}$ $(i = 1, \dots, n, j = 1, k-2)$ having degree sequences $d_v = n, d_{u_i} = 2, d_{v_i} = k-1,\ d_{w_{i,j}} = 1$. Here, the vertex v denotes the root vertex of the banana tree graph, u_i are the vertices adjacent to the root, v_i are the internal vertices belonging to each star component and adjacent to the root, and w_{ij} represent the leaf vertices of the corresponding star graphs. This is illustrated in Figure 1. The eccentricities of the respective vertices are given by $\varepsilon_G(v_i) = 5,\ \varepsilon_G(u_i) = 4,\ \varepsilon_G(v) = 3,\ \varepsilon_G(w_{ij}) = 6$.

Based on the degree of end vertices, edge set can be partitioned as

$$E_1 = \{(u,v) \mid d_u = k-1, d_v = 1\}$$

$$E_2 = \{(u,v) \mid d_u = k-1, d_v = 2\}$$

$$E_3 = \{(u,v) \mid d_u = 2,\ d_v = n\}$$

Then

$$M_1^*(B(n,k)) = \sum_{uv \in E(G)} \varepsilon_G(u) + \varepsilon_G(v)$$

$$= \sum_{uv \in E_1} \varepsilon_G(u) + \varepsilon_G(v) + \sum_{uv \in E_2} \varepsilon_G(u) + \varepsilon_G(v) + \sum_{uv \in E_3} \varepsilon_G(u) + \varepsilon_G(v)$$

$$= (6+5)n(k-2) + (4+5)n + (4+3)n = 11nk - 6n$$

$$M_1^{**}(B(n,k)) = \sum_{v\in V(G)} (\varepsilon_G(v))^2 = 6^2 n(k-2) + 5^2.n + 4^2.n + 3^2.1$$

$$= 36nk - 31n + 9$$

$$M_2^*(B(n,k)) = \sum_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v) = 6.5.n(k-2) + 4.5n + 4.3.n = 30kn - 28n.$$

New versions of Zagreb indices for the various $B(n,k)$ graphs are given in Figure 3.

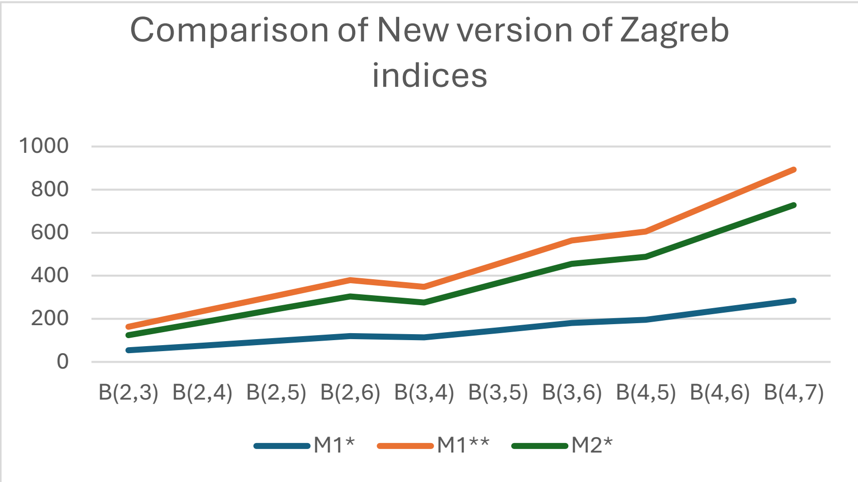


Figure 3: New version of Zagreb indices for different banana tree graphs.

The results highlight that M_1^{**} is the most sensitive index to the growth of banana tree graphs, followed by M_2^* while M_1^* shows comparatively slower growth. This indicates distinct structural responses of the three indices to variations in $B(n,k)$.

Theorem 2: For a Banana tree graph, $B(n,k)$, where $n \geq 2$ and $k \geq 3$, $n,k \in \mathbb{Z}^+$, we have the following:

$$R\big(B(n,k)\big) = n\left\{\frac{(k-2)}{\sqrt{(k-1)}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{2(k-1)}}\right\}$$

$$M\big(B(n,k)\big) = \frac{2nk-3n+k-1}{2(k-1)}$$

$$ABC(B(n,k)) = n\left(\sqrt{2} + \sqrt{\frac{k-2}{k-1}} \times (k-2)\right)$$

$$GA(B(n,k)) = 2n\left\{\frac{(k-2)\sqrt{(k-1)}}{k} + \frac{\sqrt{2n}}{2+n} + \frac{\sqrt{2(k-1)}}{k+1}\right\}$$

$$ABC_4\big(B(n,k)\big) = n\left\{(k-2)\sqrt{\frac{2k-3}{k(k-1)}} + \sqrt{\frac{2k-1}{k(k+1)}} + \sqrt{\frac{k+2n-1}{2n(k+1)}}\right\}$$

$$ABC_5\big(B(n,k)\big) = 2n\left\{\frac{(k-2)\sqrt{k(k-1)}}{2k-1} + \frac{\sqrt{k(k+1)}}{2k+1} + \frac{\sqrt{2n(k+1)}}{2n+k+1}\right\}$$

$$S\big(B(n,k)\big) = n(k-2)\left(\frac{k(k-1)}{2k-3}\right)^3 + n\left(\frac{k(k+1)}{2k-1}\right)^3 + n\left(\frac{2n(k+1)}{2n+k-1}\right)^3$$

Proof:

Consider the following edge set of a Banana tree $B(n,k)$.

$$E_1 = \{(u,v) \mid d_u = k-1, d_v = 1\}$$

$$E_2 = \{(u,v) \mid d_u = k-1, d_v = 2\}$$

$$E_3 = \{(u,v) \mid d_u = 2,\ d_v = n\}$$

Then $|E_1| = n(k-2)$, $|E_2| = n$, and $|E_3| = n$.

Then $ABC_5\big(B(n,k)\big) = 2\frac{\sqrt{k(k-1)}}{k+k-1} \times n(k-2) + 2\frac{\sqrt{k(k+1)}}{k+k+1} \times n + 2\frac{\sqrt{2n(k+1)}}{2n+k+1} \times n$

$$= 2n\left\{\frac{(k-2)\sqrt{k(k-1)}}{2k-1} + \frac{\sqrt{k(k+1)}}{2k+1} + \frac{\sqrt{2n(k+1)}}{2n+k+1}\right\}$$

$$S(B(n,k)) = \sum_{uv\in E(G)}\left(\frac{S_uS_v}{S_u+S_v-2}\right)^3 = \left(\frac{k(k-1)}{k+k-1-2}\right)^3 \times n(k-2) + \left(\frac{k(k+1)}{k+k+1-2}\right)^3 \times n + \left(\frac{2n(k+1)}{2n+k+1-2}\right)^3 \times n$$

$$= \left(\frac{k(k-1)}{2k-3}\right)^3 \times n(k-2) + \left(\frac{k(k+1)}{2k-1}\right)^3 \times n + \left(\frac{2n(k+1)}{2n+k-1}\right)^3 \times n$$

$$ABC_4\big(B(n,k)\big) = \sqrt{\frac{k-1+k-2}{k(k-1)}} \times n(k-2) + \sqrt{\frac{k+k+1-2}{k(k+1)}} \times n + \sqrt{\frac{(k+1)+2n-2}{(k+1)2n}} \times n$$

$$= n\left\{(k-2)\sqrt{\frac{2k-3}{k(k-1)}} + \sqrt{\frac{2k-1}{k(k+1)}} + \sqrt{\frac{k+2n-1}{2n(k+1)}}\right\}$$

$$R\big(B(n,k)\big) = \sum_{uv\in E(G)}\frac{1}{\sqrt{d_ud_v}} = \frac{1}{\sqrt{1.(k-1)}}n(k-2) + \frac{1}{\sqrt{2n}}n + \frac{1}{\sqrt{2(k-1)}}n$$

$$= n\left\{\frac{(k-2)}{\sqrt{(k-1)}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{2(k-1)}}\right\}$$

$$GA(B(n,k)) = \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u+d_v} = \frac{2n(k-2)\sqrt{1.(k-1)}}{1+k-1} + \frac{2n\sqrt{2n}}{2+n} + \frac{2n\sqrt{2(k-1)}}{2+k-1}$$

$$= 2n\left\{\frac{(k-2)\sqrt{(k-1)}}{k} + \frac{\sqrt{2n}}{2+n} + \frac{\sqrt{2(k-1)}}{k+1}\right\}$$

$$M\big(B(n,k)\big) = \sum_{uv\in E(G)} \frac{1}{d_u d_v} = \frac{n}{2n} + \frac{n}{2(k-1)} + \frac{(k-2)n}{k-1} = \frac{2nk-3n+k-1}{2(k-1)}$$

$$ABC(B(n,k)) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}} = \sqrt{\frac{1+k-1-2}{1.(k-1)}} \times n(k-2) + \sqrt{\frac{2+k-1-2}{2.(k-1)}} \times n + \sqrt{\frac{n+2-2}{2n}} \times n \quad = n\left(\sqrt{2} + \sqrt{\frac{k-2}{k-1}} \times (k-2)\right)$$

Figure 4 summarizes the topological index values for different banana tree graphs.

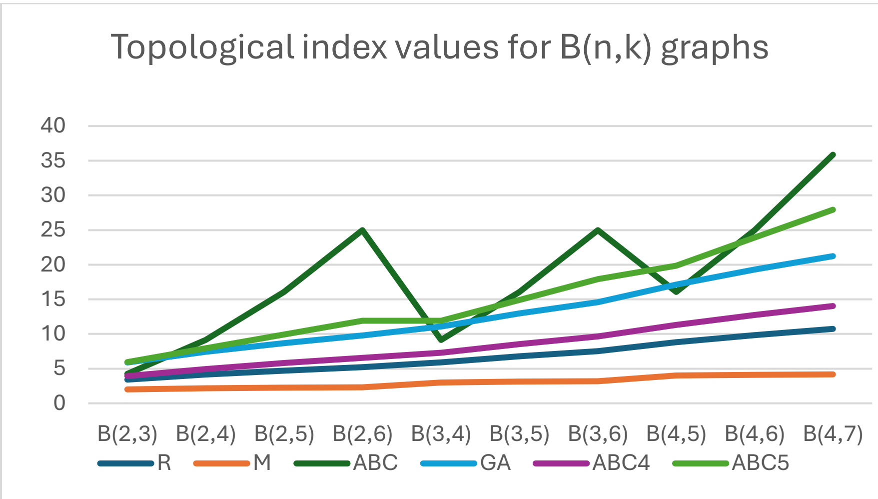
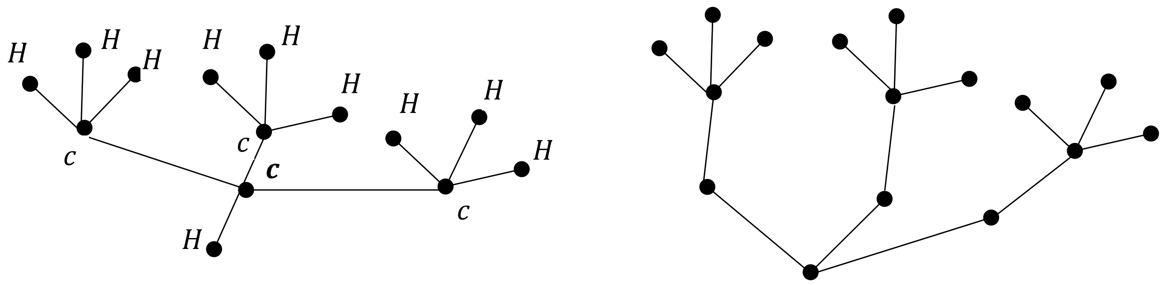


Figure 4: Topological index values for different $B(n,k)$

According to Figure 4, R, GA, and ABC_4 indices show a regular but moderate increase while R index rises more sharply compared to GA and ABC_4. The M index remains almost constant and very small across all banana trees, showing the least variation.

3.2 Interpretation in Chemical Context

When $n = 3, m = 4$, the banana tree graph resembles the carbon skeleton of a tri-branched hydrocarbon, where the root vertex corresponds to a tertiary carbon atom. Figure 5 shows both graphs of Isobutane molecule and $B(3,4)$. As the parameter n increases, the degree of the root vertex increases, representing higher branching in molecular structures. Since physicochemical properties such as boiling point and stability are influenced by branching, the computed degree-based indices can serve as structural predictors for such branched hydrocarbons.



(a) Isobutane (C_4H_{10}) (b) Banana tree graph $B(3,4)$.

Figure 5: Structural comparison between isobutane and the banana tree graph $B(3,4)$.

The banana tree graph resembles early-generation dendritic structures such as Poly(amidoamine) dendrimer, where a central core gives rise to several short branching arms [21]. Although banana trees do not capture the full recursive growth of higher-generation dendrimers, they provide a simplified mathematical model for studying branching density and terminal groups. Since many physical and chemical properties of dendrimers depend on how densely they are branched and how many terminal groups they contain, degree-based topological indices help to measure this structural complexity. Therefore, the closed-form formulas derived in this study show how changes in branching affect important molecular descriptors in dendrimer-like structures.

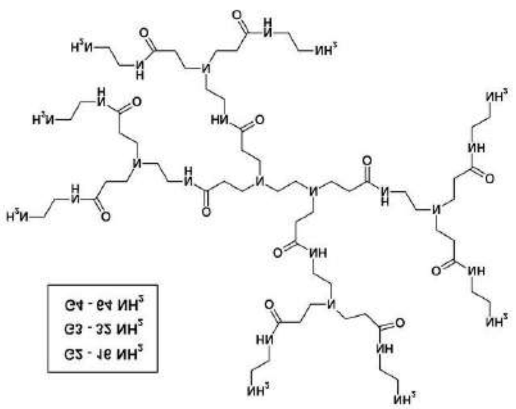


Figure 6: Chemical structure of PAMAM dendrimer Generation 1 [21]

4. CONCLUSION

In this research, closed form expressions derived for several degree based topological indices of banana tree graphs such as Randić, new versions of Zagreb indices, modified Zagreb index, geometric arithmetic index, atomic bond connectivity, fourth atomic bond connectivity and fifth atomic bond connectivity. The obtained formulas provide a general parametric representation in terms of the structural parameters of the graph, enabling direct computation without repeated structural analysis. The different indices respond differently to increases in branching. In particular, as the parameter n increases, the central vertex becomes more highly connected, leading to significant changes in the values of the indices. This demonstrates that these descriptors are sensitive to structural variation and branching density. From a chemical perspective, banana tree graphs can be viewed as simplified models of branched molecular frameworks, including tri-branched hydrocarbons and dendrimers. Since many physicochemical properties of branched organic molecules depend on their structural complexity and branching pattern, the computed indices provide useful quantitative measures for such systems. Therefore, the results of this study not only extend the theoretical understanding of degree-based topological indices for banana tree graphs but also highlight their potential relevance in chemical graph theory and the modeling of branched molecular structures.

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